Determinantal point processes associated with Hilbert spaces of holomorphic functions

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Abstract

We study determinantal point processes on $\mathbb{C}$ induced by the reproducing kernels of generalized Fock spaces as well as those on the unit disc $\mathbb{D}$ induced by the reproducing kernels of generalized Bergman spaces. In the first case, we show that all reduced Palm measures *of the same order* are equivalent. The Radon-Nikodym derivatives are computed explicitly using regularized multiplicative functionals. We also show that these determinantal point processes are rigid in the sense of Ghosh and Peres, hence reduced Palm measures *of different orders* are singular. In the second case, we show that all reduced Palm measures, *of all orders*, are equivalent. The Radon-Nikodym derivatives are computed using regularized multiplicative functionals associated with certain Blaschke products. The quasi-invariance of these determinantal point processes under the group of diffeomorphisms with compact supports follows as a corollary.

Keywords. Determinantal point processes, Palm measures, generalized Fock spaces, generalized Bergman spaces, regularized multiplicative functionals, rigidity.

Contents

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1 Introduction

1.1 Main results

1.1.1 The case of $\mathbb{C}$

Let $\psi : \mathbb{C} \to \mathbb{R}$ be a C^2-smooth function and equip the complex plane $\mathbb{C}$ with the measure $e^{-2\psi(z)} d\lambda(z)$, where $d\lambda$ is the Lebesgue measure. Assume that there exist positive constants $m, M > 0$ so that

$$m \leq \Delta\psi \leq M, \tag{1}$$

where Δ is the Euclidean Laplacian differential operator.

Denote by $\mathscr{F}_\psi$ the generalized Fock space with respect to the weight $e^{-2\psi(z)}$ and let B_ψ be the reproducing kernel of $\mathscr{F}_\psi$, whose definition is recalled in Definition 3.1. The condition (1) implies in particular the useful Christ's pointwise estimate for the reproducing kernel B_ψ, see Theorem 3.1 below.

By the Macchì-Soshnikov theorem, the kernel B_ψ induces a determinantal point process on $\mathbb{C}$, which will be denoted by $\mathbb{P}_{B_\psi}$. For more background on determinantal point processes, see, e.g. [11], [14], [21], [15] and §2 below.

Let $\mathfrak{p} \in \mathbb{C}^\ell$ and $\mathfrak{q} \in \mathbb{C}^k$ be two tuples of *distinct* points in $\mathbb{C}$. Denote by $\mathbb{P}^{\mathfrak{p}}_{B_\psi}$ and $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$ the reduced Palm measures of $\mathbb{P}_{B_\psi}$ conditioned at $\mathfrak{p}$ and $\mathfrak{q}$ respectively. For the definition, see, e.g. [12], here, we follow the notation and conventions of [1].

Our first main result is that, under the assumption (1), Palm measures $\mathbb{P}^{\mathfrak{p}}_{B_\psi}$ and $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$ of the same order are equivalent.

Theorem 1.1 (Palm measures of the same order)**.** *Let ψ satisfy* (1) *and let $\mathfrak{p}, \mathfrak{q} \in \mathbb{C}^\ell$ be any two tuples of distinct points in $\mathbb{C}$. Then*

1) The limit

$$\Sigma_{\mathfrak{p},\mathfrak{q}}(\mathcal{Z}) := \lim_{R\to\infty} \Bigg\{ \sum_{z\in\mathcal{Z}:|z|\le R} \log \left| \frac{(z-p_1)\dots(z-p_\ell)}{(z-q_1)\dots(z-q_\ell)} \right| - \mathbb{E}_{\mathbb{P}^{\mathfrak{q}}_{B_\psi}} \sum_{z\in\mathcal{Z}:|z|\le R} \log \left| \frac{(z-p_1)\dots(z-p_\ell)}{(z-q_1)\dots(z-q_\ell)} \right| \Bigg\}$$

exists for $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$-almost every configuration $\mathcal{Z}$ and the function $\mathcal{Z} \to e^{2\Sigma_{\mathfrak{p},\mathfrak{q}}(\mathcal{Z})}$ is integrable with respect to $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$.

2) The Palm measures $\mathbb{P}^{\mathfrak{p}}_{B_\psi}$ and $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$ are equivalent. Moreover, for $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$-almost every configuration $\mathcal{Z}$, we have

$$\frac{d\mathbb{P}^{\mathfrak{p}}_{B_\psi}}{d\mathbb{P}^{\mathfrak{q}}_{B_\psi}}(\mathcal{Z}) = \frac{e^{2\Sigma_{\mathfrak{p},\mathfrak{q}}(\mathcal{Z})}}{\mathbb{E}_{\mathbb{P}^{\mathfrak{q}}_{B_\psi}}(e^{2\Sigma_{\mathfrak{p},\mathfrak{q}}})}. \tag{2}$$

Definition 1.1 (Ghosh [8], Ghosh-Peres[9])**.** A point process $\mathbb{P}$ on $\mathbb{C}$ is said to be rigid if for any bounded open set $D \subset \mathbb{C}$ with Lebesgue-negligible boundary ∂D, there exists a function F_D defined on the set of configurations, measurable with respect to the σ-algebra generated by the family of random variables $\{\#_A : A \subset \mathbb{C} \setminus D \text{ bounded and Borel}\}$, where $\#_A$ is defined by

$$\#_A(\mathcal{Z}) = \text{ the cardinality of the finite set } \mathcal{Z} \cap A,$$

such that

$$\#_D(\mathcal{Z}) = F_D(\mathcal{Z} \setminus D), \text{ for } \mathbb{P}\text{-almost every configuration } \mathcal{Z} \text{ over } \mathbb{C}.$$

Proposition 1.2 (Rigidity)**.** *Under the assumption* (1)*, the determinantal point process* $\mathbb{P}_{B_\psi}$ *is rigid in the sense of Ghosh and Peres.*

Proposition 8.1 in the Appendix now implies

Corollary 1.3 (Palm measures of different orders)**.** *Under the assumption* (1)*, if* $\ell \neq k$*, then the reduced Palm measures* $\mathbb{P}_{B_\psi}^{\mathfrak{p}}$ *and* $\mathbb{P}_{B_\psi}^{\mathfrak{q}}$ *are mutually singular.*

Remark 1.1. In the particular case $\psi(z) = \frac{1}{2}|z|^2$ (Ginibre point process), the results of Theorem 1.1 and Corollary 1.3 were obtained in [17] with a different approach, where the authors used finite dimensional approximation by orthogonal polynomial ensembles. The rigidity in the case $\psi(z) = \frac{1}{2}|z|^2$ is due to Ghosh and Peres [9], their original approach will be followed in our proof of Proposition 1.2.

1.1.2 The case of $\mathbb{D}$

In the case of Bergman spaces on the unit disc $\mathbb{D}$, the situation becomes quite different and the corresponding determinantal point processes in this case are not rigid.

Consider a weight function $\omega : \mathbb{D} \to \mathbb{R}^+$ and equip $\mathbb{D}$ with the measure $\omega(z)d\lambda(z)$. Assume that ω satisfies that

$$\int_{\mathbb{D}} (1-|z|)^2 B_\omega(z,z)\omega(z)d\lambda(z) < \infty. \tag{3}$$

We will denote by $\mathscr{B}_\omega$ the generalized Bergman space on $\mathbb{D}$ with respect to the weight ω, and by B_ω its reproducing kernel, the definition is recalled in Definition 3.2.

Again, by the Macchì-Soshnikov theorem, the reproducing kernel B_ω induces a determinantal point process on $\mathbb{D}$, which we denote by $\mathbb{P}_{B_\omega}$.

Let $\mathfrak{p} \in \mathbb{D}^\ell$ be an ℓ-tuple of distinct points in $\mathbb{D}$ and denote by $\mathbb{P}_{B_\omega}^{\mathfrak{p}}$ the reduced Palm measures of $\mathbb{P}_{B_\omega}$ at $\mathfrak{p}$.

Under the assumption (3), we show, for any $\mathfrak{p} \in \mathbb{D}^\ell$ of distinct points in $\mathbb{D}$, the reduced Palm measure $\mathbb{P}_{B_\omega}^{\mathfrak{p}}$ is equivalent to $\mathbb{P}_{B_\omega}$. In particular, any two reduced Palm measures are equivalent. For the weight $\omega \equiv 1$, this result is due to Holroyd and Soo [10].

We now proceed to the statement of our main result in the case of $\mathbb{D}$. For an ℓ-tuple $\mathfrak{p} = (p_1, \cdots, p_\ell)$ of distinct points in $\mathbb{D}$, set

$$b_{\mathfrak{p}}(z) = \prod_{j=1}^{\ell} \frac{z - p_j}{1 - \bar{p}_j z}. \tag{4}$$

Theorem 1.4. *Let* ω *be a weight such that* (3) *holds. Let* $\mathfrak{p} \in \mathbb{D}^\ell$ *be an* ℓ*-tuple of distinct points in* $\mathbb{D}$*. Then*

1) The limit

$$S_{\mathfrak{p}}(\mathcal{Z}) := \lim_{r\to 1^-}\left(\sum_{z\in\mathcal{Z}:|z|\le r}\log|b_{\mathfrak{p}}(z)| - \mathbb{E}_{\mathbb{P}_{B_\omega}}\sum_{z\in\mathcal{Z}:|z|\le r}\log|b_{\mathfrak{p}}(z)|\right) \tag{5}$$

exists for $\mathbb{P}_{B_\omega}$-almost every configuration $\mathcal{Z}$ and the function $\mathcal{Z}\to e^{2S_{\mathfrak{p}}(\mathcal{Z})}$ is integrable with respect to $\mathbb{P}_{B_\omega}$.

2) The Radon-Nikodym derivative $d\mathbb{P}^{\mathfrak{p}}_{B_\omega}/d\mathbb{P}_{B_\omega}$ is given by the formula:

$$\frac{d\mathbb{P}^{\mathfrak{p}}_{B_\omega}}{d\mathbb{P}_{B_\omega}}(\mathcal{Z}) = \frac{e^{2S_{\mathfrak{p}}(\mathcal{Z})}}{\mathbb{E}_{\mathbb{P}_{B_\omega}}(e^{2S_{\mathfrak{p}}})}, \text{ for } \mathbb{P}_{B_\omega}\text{-almost every configuration } \mathcal{Z}. \tag{6}$$

Theorem 1.4 will be obtained from

Proposition 1.5. *Let ω be a weight such that (3) holds. Let $\mathfrak{p}\in\mathbb{D}^\ell$ and $\mathfrak{q}\in\mathbb{D}^k$ be two tuples of distinct points in $\mathbb{D}$. Then the Radon-Nikodym derivative $d\mathbb{P}^{\mathfrak{p}}_{B_\omega}/d\mathbb{P}^{\mathfrak{q}}_{B_\omega}$ is given by*

$$\frac{d\mathbb{P}^{\mathfrak{p}}_{B_\omega}}{d\mathbb{P}^{\mathfrak{q}}_{B_\omega}}(\mathcal{Z}) = \frac{e^{2S_{\mathfrak{p},\mathfrak{q}}(\mathcal{Z})}}{\mathbb{E}_{\mathbb{P}^{\mathfrak{q}}_{B_\omega}}(e^{2S_{\mathfrak{p},\mathfrak{q}}})}, \text{ for } \mathbb{P}^{\mathfrak{q}}_{B_\omega}\text{-almost every configuration } \mathcal{Z}, \tag{7}$$

where $S_{\mathfrak{p},\mathfrak{q}}(\mathcal{Z})$ is defined for $\mathbb{P}^{\mathfrak{q}}_{B_\omega}$-almost every configuration $\mathcal{Z}$, given by

$$S_{\mathfrak{p},\mathfrak{q}}(\mathcal{Z}) := \lim_{r\to 1^-}\left(\sum_{z\in\mathcal{Z}:|z|\le r}\log|b_{\mathfrak{p}}(z)b_{\mathfrak{q}}(z)^{-1}| - \mathbb{E}_{\mathbb{P}^{\mathfrak{q}}_{B_\omega}}\sum_{z\in\mathcal{Z}:|z|\le r}\log|b_{\mathfrak{p}}(z)b_{\mathfrak{q}}(z)^{-1}|\right). \tag{8}$$

Remark 1.2. If ψ (resp. ω) is a radial function, then the monomials $(z^n)_{n\ge 0}$ are orthogonal in the corresponding Hilbert space, hence the determinantal point process $\mathbb{P}_{B_\psi}$ (resp. $\mathbb{P}_{B_\omega}$) can be naturally approximated by *orthogonal polynomial ensembles*. In particular, if $\psi(z) = \frac{1}{2}|z|^2$ for all $z\in\mathbb{C}$, then $\mathbb{P}_{B_\psi}$ is the Ginibre point process, see chapter 15 of Mehta's book [16]; if $\omega(z)\equiv 1$ for all $z\in\mathbb{D}$, then $\mathbb{P}_{B_\omega}$ is the determinantal point process describing the zero set of a Gaussian analytic function on the hyperbolic disc $\mathbb{D}$, see [18]. Our study, however, goes beyond the radial setting and our methods work for more general phase spaces as well.

Remark 1.3. The regularized multiplicative functionals are necessary in Theorem 1.1, Theorem 1.4 and Proposition 1.5: indeed, when $\omega\equiv 1$, for $\mathbb{P}_{B_\omega}$-almost every configuration $\mathcal{Z}$ on $\mathbb{D}$, the points in the configuration $\mathcal{Z}$ violate the Blaschke condition:

$$\sum_{z\in\mathcal{Z}}(1-|z|) = \infty, \tag{9}$$

whence for any $\mathfrak{p}\in\mathbb{D}^\ell$, we have,

$$\prod_{z\in\mathcal{Z}}|b_{\mathfrak{p}}(z)| = 0, \text{ for } \mathbb{P}_{B_\omega}\text{-almost every configuration } \mathcal{Z}, \tag{10}$$

so the simple multiplicative functional is identically 0. To see (9), we use the Kolmogorov three-series theorem and the fact (Peres and Virág [18]) that, for $\mathbb{P}_{B_\omega}$-distributed random configurations $\mathcal{Z}$, the set of moduli $\{|z| : z \in \mathcal{Z}\}$ has same law as the set of random variables $\{U_k^{1/(2k)}\}$, where $U_1, U_2, \dots$ are independent identically distributed random variables such that U_1 has a uniform distribution in $[0,1]$. A direct computation shows that

$$\mathbb{E}_{\mathbb{P}_{B_\omega}} \sum_{z \in \mathcal{Z}} (1 - |z|) = \sum_k (1 - \mathbb{E}(U_k^{1/(2k)})) = \infty.$$

The determinantal point process $\mathbb{P}_{B_\omega}$ in the case $\omega \equiv 1$ describes the zero set of a Gaussian analytic function on $\mathbb{D}$:

$$F_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} g_n z^n,$$

where $(g_n)_{n\geq 0}$ is a sequence of independent identically distributed standard complex Gaussian random variables. Direct computation shows that

$$\mathbb{E}\|F_{\mathbb{D}}\|_{H^2}^2 = \infty \text{ and } \mathbb{E}\|F_{\mathbb{D}}\|_{B_\omega}^2 = \infty,$$

hence the random holomorphic function almost surely belongs neither to the Hardy space H^2 nor to the Bergman space, thus it is not surprising that the zero set of $F_{\mathbb{D}}$ almost surely violates Blaschke condition.

1.2 Quasi-invariance

Let $U = \mathbb{C}$ or $\mathbb{D}$. Let $F : U \to U$ be a diffeomorphism. Its support, denoted by $\mathrm{supp}(F)$, is defined as the *relative closure* in U of the subset $\{z \in U : F(z) \neq z\}$. The totality of diffeomorphisms with compact supports is a group denoted by $\mathrm{Diff}_c(U)$, i.e.,

$$\mathrm{Diff}_c(U) := \left\{ F : U \to U \,\middle|\, F \text{ is a diffeomorphism and } \mathrm{supp}(F) \text{ is compact} \right\}.$$

The group $\mathrm{Diff}_c(U)$ naturally acts on the set of configurations on U: given any diffeomorphism $F \in \mathrm{Diff}_c(U)$ and any configuration $\mathcal{Z}$ on U,

$$(F, \mathcal{Z}) \mapsto F(\mathcal{Z}) := \{F(z) : z \in \mathcal{Z}\}.$$

Recall that the Jacobian J_F of the function $F : U \to U$ is defined by

$$J_F(z) = |\det DF(z)|.$$

Corollary 1.6. *Let $\mathbb{P}_K$ be a determinantal point process on U, which is either the determinantal point process $\mathbb{P}_{B_\psi}$ on $\mathbb{C}$ or the determinantal point process $\mathbb{P}_{B_\omega}$ on $\mathbb{D}$. Then under*

Assumption (1) *in the case of* $\mathbb{C}$ *or, in the case of* $\mathbb{D}$ *Assumption* (3), $\mathbb{P}_K$ *is quasi-invariant under the induced action of the group* $\mathrm{Diff}_c(U)$.

More precisely, let $F \in \mathrm{Diff}_c(U)$ *and let* $V \subset U$ *be any precompact subset containing* $\mathrm{supp}(F)$. *For* $\mathbb{P}_K$*-almost every configuration* $\mathscr{Z}$ *the following holds: if* $\mathscr{Z} \bigcap V = \{q_1, \dots, q_\ell\}$, *then*

$$\frac{d\mathbb{P}_K \circ F}{d\mathbb{P}_K}(\mathscr{Z}) = \frac{\det[K(F(q_i), F(q_j))]_{i,j=1}^{\ell}}{\det[K(q_i, q_j)]_{i,j=1}^{\ell}} \cdot \frac{d\mathbb{P}_K^{\mathfrak{p}}}{d\mathbb{P}_K^{\mathfrak{q}}}(\mathscr{Z}) \cdot \prod_{i=1}^{\ell} J_F(q_i),$$

where $\mathfrak{q} = (q_1, \dots, q_\ell) \in U^\ell$ *and* $\mathfrak{p} = (F(q_1), \dots, F(q_\ell)) \in U^\ell$

Proof. This is an immediate consequence of Theorem 1.1, Proposition 1.5 and Proposition 2.9 of [1]. □

1.3 Unified approach for obtaining Radon-Nikodym derivatives

In this section, let us describe briefly the main idea of our unified approach for obtaining the Radon-Nikodym derivatives in Theorem 1.1, Theorem 1.4 and Proposition 1.5.

1.3.1 Relations between Palm subspaces

If $\mathfrak{p} \in \mathbb{C}^\ell$ is an ℓ-tuple of distinct points of $\mathbb{C}$, we define the *Palm subspace*:

$$\mathscr{F}_\psi(\mathfrak{p}) := \{\varphi \in \mathscr{F}_\psi : \varphi(p_1) = \cdots = \varphi(p_\ell) = 0\}\,. \tag{11}$$

Let $B_\psi^{\mathfrak{p}}$ denote the reproducing kernel of $\mathscr{F}_\psi(\mathfrak{p})$.

Similarly, if $\mathfrak{p} \in \mathbb{D}^\ell$ is an ℓ-tuple of distinct points of $\mathbb{D}$, we define the Palm subspace

$$\mathscr{B}_\omega(\mathfrak{p}) = \{\varphi \in \mathscr{B}_\omega : \varphi(p_1) = \cdots = \varphi(p_\ell) = 0\}\,, \tag{12}$$

and denote its reproducing kernel by $B_\omega^{\mathfrak{p}}$.

By Shirai-Takahashi's theorem, which motivates our terminology, see Theorem 2.1 below, these Palm subspaces are related to the reduced Palm measures: $B_\psi^{\mathfrak{p}}$ (resp. $B_\omega^{\mathfrak{p}}$) is the correlation kernel of $\mathbb{P}_{B_\psi}^{\mathfrak{p}}$ (resp. $\mathbb{P}_{B_\omega}^{\mathfrak{p}}$), i.e., we have

$$\mathbb{P}_{B_\psi}^{\mathfrak{p}} = \mathbb{P}_{B_\psi^{\mathfrak{p}}} \text{ (resp. } \mathbb{P}_{B_\omega}^{\mathfrak{p}} = \mathbb{P}_{B_\omega^{\mathfrak{p}}}).$$

Proposition 1.7. *For any pair of* ℓ*-tuples* $\mathfrak{p}, \mathfrak{q} \in \mathbb{C}^\ell$ *of distinct points in* $\mathbb{C}$, *we have*

$$\mathscr{F}_\psi(\mathfrak{p}) = \frac{(z-p_1)\cdots(z-p_\ell)}{(z-q_1)\cdots(z-q_\ell)} \cdot \mathscr{F}_\psi(\mathfrak{q}). \tag{13}$$

Proposition 1.8. *Let $k, \ell \in \mathbb{N} \cup \{0\}$ and let $\mathfrak{p} \in \mathbb{D}^\ell, \mathfrak{q} \in \mathbb{D}^k$ be two tuples of distinct points in $\mathbb{D}$, then*

$$\mathscr{B}_\omega(\mathfrak{p}) = \prod_{j=1}^{\ell} \frac{z - p_j}{1 - \bar{p}_j z} \left(\prod_{j=1}^{k} \frac{z - q_j}{1 - \bar{q}_j z} \right)^{-1} \cdot \mathscr{B}_\omega(\mathfrak{q}). \tag{14}$$

In particular, we have

$$\mathscr{B}_\omega(\mathfrak{p}) = \prod_{j=1}^{\ell} \frac{z - p_j}{1 - \bar{p}_j z} \cdot \mathscr{B}_\omega.$$

Comments.
- The proofs of Propositions 1.7 and 1.8 are immediate from the definitions (11) and (12) and basic properties of holomorphic functions.
- Notice the analogy of the above Propositions 1.7 and 1.8 with Proposition 3.4 in [1].
- A common feature, which is crucially used later, of Propositions 1.7 and 1.8, is the following relations

$$\lim_{|z| \to \infty} \left| \frac{(z - p_1) \cdots (z - p_\ell)}{(z - q_1) \cdots (z - q_\ell)} \right| = 1 \text{ and } \lim_{|z| \to 1^-} \left| \prod_{j=1}^{\ell} \frac{z - p_j}{1 - \bar{p}_j z} \right| = 1. \tag{15}$$

The rate of convergence in (15) also plays an important rôle for defining the regularized multiplicative functionals, see §5.2 and §6.2.

1.3.2 Radon-Nikodym derivatives as regularized multiplicative functionals

For obtaining the Radon-Nikodym derivatives in question, we will first develop in Theorem 4.1, the most technical result of this paper, a general method on regularized multiplicative functionals. This result, an extension of Proposition 4.6 of [1], is, we hope, interesting in its own right; the stronger statement is also necessary for our argument in the case of $\mathbb{C}$, in which Proposition 4.6 of [1] is not applicable.

By Theorem 4.1, under the assumption (1) on ψ, we can show that the regularized multiplicative functional, i.e., the formula (7), is well-defined. This regularized multiplicative functional is then shown to be exactly the Radon-Nikodym derivative between the desired reduced Palm measures of the same order for the determinantal point process $\mathbb{P}_{B_\psi}$.

The regularized multiplicative functionals in the case of $\mathbb{D}$ are technically simpler and the full force of Theorem 4.1 is not needed.

1.4 Organization of the paper

The paper is organized as follows. In the introduction section §1, we give necessary definitions and notation and state our main results. The basic materials in the theory of determinantal point processes are recalled in §2. The definitioins concerning generalized Fock spaces and generalized Bergman spaces are given in §3. In §4, our main ingredient, *regularized multiplicative functionals*, is defined. We also state the most technical Theorem 4.1 in §4. We then apply Theorem 4.1 to prove our main results for determinantal point processes associated with generalized Fock spaces in §5 and to prove the main results in the case of generalized Bergman spaces in §6. The section §7 is devoted to the proof of Theorem 4.1. In the Appendix §8, we give details for the fact that rigid point processes have singular Palm measures with different orders.

Remark 1.4. Part of our main results in this paper were announced in [3].

2 Spaces of configurations and determinantal point processes

For the reader's convenience, we recall the basic definitions and notation on determinantal point processes.

Let E be a locally compact complete separable metric space equipped with a sigma-finite Borel measure μ. The space E will be later referred to as *phase space*. The measure μ is referred to as *reference measure* or *background measure*. By a configuration $\mathcal{X}$ on the phase space E, we mean a locally finite subset of $\mathcal{X} \subset E$. By identifying any configuration $\mathcal{X} \in \mathrm{Conf}(E)$ with the Radon measure

$$m_{\mathcal{X}} := \sum_{x \in \mathcal{X}} \delta_x,$$

where δ_x is the Dirac mass on the point x, the space of configurations $\mathrm{Conf}(E)$ is identified with a subset of the space $\mathfrak{M}(E)$ of Radon measures on E and becomes itself a complete separable metric space. We equip $\mathrm{Conf}(E)$ with its Borel sigma algebra.

Points in a configuration will also be called particles. In this paper, the italicized letters as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ always denote configurations.

2.1 Additive functionals and multiplicative functionals

We recall the definitions of additive and multiplicative functionals on the space of configurations.

If $\varphi : E \to \mathbb{C}$ is a measurable function on E, then the additive functional (which is also called linear statistic) $S_\varphi : \mathrm{Conf}(E) \to \mathbb{C}$ corresponding to φ is defined by

$$S_\varphi(\mathcal{X}) = \sum_{x \in \mathcal{X}} \varphi(x)$$

provided the sum $\sum_{x \in \mathcal{X}} \varphi(x)$ converges absolutely. If the sum $\sum_{x \in \mathcal{X}} \varphi(x)$ fails to converge absolutely, then the additive functional is not defined at $\mathcal{X}$.

Similarly, the multiplicative functional $\Psi_g : \mathrm{Conf}(E) \to [0, \infty]$ associated with a non-negative measurable function $g : E \to \mathbb{R}^+$, is defined as the function

$$\Psi_g(\mathcal{X}) := \prod_{x \in \mathcal{X}} g(x),$$

provided the product $\prod\limits_{x \in \mathcal{X}} g(x)$ absolutely converges to a value in $[0, \infty]$. If the product $\prod\limits_{x \in \mathcal{X}} g(x)$ fails to converge absolutely, then the multiplicative functional is not defined at the configuration $\mathcal{X}$.

2.2 Locally trace class operators and their kernels

Let $L^2(E, \mu)$ denote the complex Hilbert space of $\mathbb{C}$-valued square integrable functions on E. Let $\mathscr{S}_1(E, \mu)$ be the space of trace class operators on $L^2(E, \mu)$ equipped with the trace class norm $\| \cdot \|_{\mathscr{S}_1}$. Let $\mathscr{S}_{1,\mathrm{loc}}(E, \mu)$ be the space of locally trace class operators, that is, the space of bounded operators $K : L^2(E, \mu) \to L^2(E, \mu)$ such that for any bounded subset $B \subset E$, we have

$$\chi_B K \chi_B \in \mathscr{S}_1(E, \mu).$$

A locally trace class operator K admits a kernel, for which we use the same symbol K. In this paper, we are especially interested in locally trace class orthogonal projection operators. Let, therefore, $\Pi \in \mathscr{S}_{1,\mathrm{loc}}$ be an operator of orthogonal projection onto a closed subspace $L \subset L^2(E, \mu)$. All kernels considered in this paper are supposed to satisfy the following

Assumption 1. There exists a subset $\widetilde{E} \subset E$, satisfying $\mu(E \setminus \widetilde{E}) = 0$ such that

- For any $q \in \widetilde{E}$, the function $v_q(\cdot) = \Pi(\cdot, q)$ lies in $L^2(E, \mu)$ and for any $f \in L^2(E, \mu)$, we have
$$(\Pi f)(q) = \langle f, v_q \rangle_{L^2(E,\mu)}.$$
In particular, if f is a function in L, then by letting $f(q) = \langle f, v_q \rangle_{L^2(E,\mu)}$, for any $q \in \widetilde{E}$, the function f is defined everywhere on $\widetilde{E}$ (which is slightly stronger than almost everywhere defined on E).

- The diagonal values $\Pi(q, q)$ of the kernel Π are defined for all $q \in \widetilde{E}$ and we have $\Pi(q, q) = \langle v_q, v_q \rangle_{L^2(E,\mu)}$. Moreover, for any bounded Borel subset $B \subset E$,
$$\mathrm{tr}(\chi_B \Pi \chi_B) = \int\limits_B \Pi(x, x) d\mu(x).$$

2.3 Definition of determinantal point processes

A Borel probability $\mathbb{P}$ on $\mathrm{Conf}(E)$ will be called a point process on E. Recall that the point process $\mathbb{P}$ is said to admit k-th correlation measure ρ_k on E^k if for any continuous compactly supported function $\varphi : E^k \to \mathbb{C}$, we have

$$\int\limits_{\mathrm{Conf}(E)} \sideset{}{^*}\sum_{x_1,\ldots,x_k\in\mathcal{X}} \varphi(x_1,\ldots,x_k)\mathbb{P}(d\mathcal{X}) = \int\limits_{E^k} \varphi(q_1,\ldots,q_k)d\rho_k(q_1,\ldots,q_k),$$

where $\sum^*$ denotes the sum over all ordered k-tuples of *distinct* points $(x_1,\ldots,x_k) \in \mathcal{X}^k$.

Given a bounded measurable subset $A \subset E$, we define $\#_A : \mathrm{Conf}(E) \to \mathbb{N} \cup \{0\}$ by

$$\#_A(\mathcal{X}) = \text{ the number of particles in } \mathcal{X} \cap A.$$

Then the point process $\mathbb{P}$ is determined by the joint distributions of $\#_{A_1},\ldots,\#_{A_n}$, if $A_1,\ldots,A_n$ range over the family of bounded measurable subsets of E.

A Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is called determinantal if there exists an operator $K \in \mathscr{S}_{1,\mathrm{loc}}(E,\mu)$ such that for any bounded measurable function g, for which $g-1$ is supported in a bounded set B, we have

$$\mathbb{E}_{\mathbb{P}}\Psi_g = \det\left(1 + (g-1)K\chi_B\right). \tag{16}$$

The Fredholm determinant is well-defined since $(g-1)K\chi_B \in \mathscr{S}_1(E,\mu)$. The equation (16) determines the measure $\mathbb{P}$ uniquely and we will denote it by $\mathbb{P}_K$ and the kernel K is said to be *a correlation kernel* of the determinantal point process $\mathbb{P}_K$. Note that $\mathbb{P}_K$ is uniquely determined by K, but different kernels may yield the same point process.

By the Macchì-Soshnikov theorem [15], [21], any Hermitian positive contraction in $\mathscr{S}_{1,\mathrm{loc}}(E,\mu)$ defines a determinantal point process. In particular, the projection operator on a *reproducing kernel Hilbert space* induces a determinantal point process.

Remark 2.1. If $\alpha : E \to \mathbb{C}$ is a Borel function such that $|\alpha(x)| = 1$ for μ-almost every $x \in E$, and if $\Pi \in \mathscr{S}_{1,\mathrm{loc}}$ is the operator of orthogonal projection onto a closed subspace $L \subset L^2(E,\mu)$, then Π and $\alpha\Pi\overline{\alpha}$ define the same determinantal point process, i.e.,

$$\mathbb{P}_{\alpha\Pi\overline{\alpha}} = \mathbb{P}_\Pi.$$

Note that $\alpha\Pi\overline{\alpha}$ is the orthogonal projection onto the subspace $\alpha(x)L$.

2.4 Palm measures and Palm subspaces

In this paper, by Palm measures, we always mean *reduced* Palm measures. We refer to [12], [5] for more details on Palm measures of general point processes.

Let $\mathbb{P}$ be a point process on $\mathrm{Conf}(E)$. Assume that $\mathbb{P}$ admits k-th correlation measure ρ_k on E^k. Then for ρ_k-almost every $\mathfrak{q} = (q_1, \dots, q_k) \in E^k$ of distinct points in E, one can define a point process on E, denoted by $\mathbb{P}^{\mathfrak{q}}$ and is called (reduced) Palm measure of $\mathbb{P}$ conditioned at $\mathfrak{q}$, by the following disintegration formula: for any non-negative Borel test function $u : \mathrm{Conf}(E) \times E^k \to \mathbb{R}$,

$$\int\limits_{\mathrm{Conf}(E)} \sum_{q_1,\dots,q_k \in \mathcal{X}}^{*} u(\mathcal{X}; \mathfrak{q}) \mathbb{P}(d\mathcal{X}) = \int\limits_{E^k} \rho_k(d\mathfrak{q}) \int\limits_{\mathrm{Conf}(E)} u(\mathcal{X} \cup \{q_1, \dots, q_k\}; \mathfrak{q}) \mathbb{P}^{\mathfrak{q}}(d\mathcal{X}), \quad (17)$$

where $\sum^{*}$ denotes the sum over all mutually distinct points $q_1, \dots, q_k \in \mathcal{X}$.

Informally, $\mathbb{P}^{\mathfrak{q}}$ is the conditional distribution of $\mathcal{X} \setminus \{q_1, \dots, q_k\}$ on $\mathrm{Conf}(E)$ conditioned to the event that all particles $q_1, \dots, q_k$ are in the configuration $\mathcal{X}$, providing that $\mathcal{X}$ has as distribution $\mathbb{P}$.

Now let $\mathbb{P}_\Pi$ be a determinantal point process on $\mathrm{Conf}(E)$ induced by the projection operator Π. Let $\mathfrak{q} = (q_1, \dots, q_k) \in \widetilde{E}^k$ be a k-tuple of distinct points in $\widetilde{E} \subset E$, where $\widetilde{E}$ is as in Assumption 1. Set

$$L(\mathfrak{q}) = \{\varphi \in L : \varphi(q_1) = \cdots = \varphi(q_k) = 0\}. \quad (18)$$

The space $L(\mathfrak{q})$ will be called the *Palm subspace* of $L^2(E, \mu)$ corresponding to $\mathfrak{q}$. Both the operator of orthogonal projection from $L^2(E, \mu)$ onto the subspace $L(\mathfrak{q})$ and the reproducing kernel of $L(\mathfrak{q})$ will be denoted by $\Pi^{\mathfrak{q}}$.

Explicit formulae for $\Pi^{\mathfrak{q}}$ in terms of the kernel Π are known, see Shirai-Takahashi [20]. Here we recall that for a single point $q \in \widetilde{E}$, we have

$$\Pi^q(x, y) = \Pi(x, y) - \frac{\Pi(x, q)\Pi(q, y)}{\Pi(q, q)}. \quad (19)$$

If $\Pi(q, q) = 0$, we set $\Pi^q = \Pi$. In general, we have the iteration

$$\Pi^{\mathfrak{q}} = (\cdots (\Pi^{q_1})^{q_2} \cdots)^{q_k}.$$

Note that the order of the points $q_1, q_2, \cdots q_k$ has no effect in the above iteration.

Theorem 2.1 (Shirai and Takahashi [20]). *For any $k \in \mathbb{N}$ and for ρ_k-almost every k-tuple $\mathfrak{q} \in E^k$ of distinct points in E, the Palm measure $\mathbb{P}_\Pi^{\mathfrak{q}}$ is induced by the kernel $\Pi^{\mathfrak{q}}$:*

$$\mathbb{P}_\Pi^{\mathfrak{q}} = \mathbb{P}_{\Pi^{\mathfrak{q}}}.$$

2.5 Rigidity

Let $\mathbb{P}$ be a point process over $\mathbb{C}$. We will use the following result on the rigidity of point processes (see Definition 1.1).

Theorem 2.2 (Ghosh [8], Ghosh and Peres [9])**.** *Let $\mathbb{P}$ be a point process on $\mathbb{C}$ whose first correlation measure ρ_1 is absolutely continuous with respect to the Lebesgue measure and let $\mathcal{D}$ be an open bounded set $\mathcal{D}$ with Lebesgue-negligible boundary. Let φ be a continuous function on $\mathbb{C}$. Suppose that for any $0 < \varepsilon < 1$, there exists a C_c^2-smooth function Φ_ε such that $\Phi_\varepsilon = \varphi$ on $\mathcal{D}$, and $\mathrm{Var}_{\mathbb{P}}(S_{\Phi_\varepsilon}) < \varepsilon$. Then the point process $\mathbb{P}$ is rigid.*

3 Generalized Fock spaces and Bergman spaces

Let $\psi : \mathbb{C} \to \mathbb{R}$ be a function satisfying the assumption (1) and denote

$$dv_\psi(z) = e^{-2\psi(z)} d\lambda(z),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Let $\mathscr{O}(\mathbb{C})$ denote the space of holomorphic functions on $\mathbb{C}$.

Definition 3.1. If the linear subspace

$$\mathscr{F}_\psi := L^2(\mathbb{C}, dv_\psi) \cap \mathscr{O}(\mathbb{C})$$

is closed in $L^2(\mathbb{C}, dv_\psi)$, then it will called generalized Fock space with respect to the measure dv_ψ. The orthogonal projection $P : L^2(dv_\psi) \to \mathscr{F}_\psi$ is given by integration against a reproducing kernel $B_\psi(z, w)$ (analytic in z and anti-analytic in w):

$$(Pf)(z) = \int_{\mathbb{C}} f(w) B_\psi(z, w) e^{-2\psi(w)} d\lambda(w). \tag{20}$$

Definition 3.2. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc. A weight function $\omega : \mathbb{D} \to \mathbb{R}^+$ is called a *Bergman weight*, if it is integrable with respect to the Lebesgue measure and the generalized Bergman space

$$\mathscr{B}_\omega := L^2(\mathbb{D}, \omega d\lambda) \cap \mathscr{O}(\mathbb{D})$$

is closed and the evaluation functionals $f \to f(z)$ on $\mathscr{B}_\omega$ are uniformly bounded on any compact subset of $\mathbb{D}$. In such situation, the space $\mathscr{B}_\omega$ is a reproducing kernel Hilbert space, its reproducing kernel will be denoted as B_ω.

We shall need Christ's pointwise estimate (cf. [4], [6], [19]) of the reproducing kernel $B_\psi(z, w)$. Theorem 3.2 in [19] gives the estimate in the form most convenient for us.

Theorem 3.1 (Christ)**.** *Let $\psi \in C^2(\mathbb{C})$ be a real-valued function satisfying* (1)*. Then there are contants $\delta, C > 0$ such that for all $z, w \in \mathbb{C}$,*

$$|B_\psi(z, w)|^2 e^{-2\psi(z) - 2\psi(w)} \leq C e^{-\delta |z - w|}. \tag{21}$$

In particular, for all $z \in \mathbb{C}$,

$$B_\psi(z, z) e^{-2\psi(z)} \leq C. \tag{22}$$

Remark 3.1. For the Gaussian case $\psi(z) = \frac{1}{2}|z|^2$, we have the following explicit formula

$$|B_\psi(z,w)|^2 e^{-2\psi(z)-2\psi(w)} = \pi^{-2} e^{-|z-w|^2}.$$

4 Regularized multiplicative functionals

As (10) shows, simple multiplicative functionals cannot be used in our situation. Following [1], we use regularized multiplicative functionals whose definition we now recall.

Let $f : E \to \mathbb{C}$ be a Borel function. Set

$$\mathrm{Var}(\Pi, f) = \frac{1}{2} \iint_{E^2} |f(x) - f(y)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y). \tag{23}$$

Introduce the Hilbert space $\mathcal{V}(\Pi)$ in the following way: the elements of $\mathcal{V}(\Pi)$ are functions f on E satisfying $\mathrm{Var}(\Pi, f) < \infty$; functions that differ by a constant are identified. The square of the norm of an element $f \in \mathcal{V}(\Pi)$ is precisely $\mathrm{Var}(\Pi, f)$.

Let $S_f : \mathrm{Conf}(E) \to \mathbb{C}$ to be the corresponding additive functional, such that $S_f \in L^1(\mathrm{Conf}(E), \mathbb{P}_\Pi)$, then we set

$$\overline{S}_f = S_f - \mathbb{E}_{\mathbb{P}_\Pi} S_f. \tag{24}$$

If moreover, $S_f \in L^2(\mathrm{Conf}(E), \mathbb{P}_\Pi)$, then it is easy to see that

$$\mathbb{E}_{\mathbb{P}_\Pi} |\overline{S}_f|^2 = \mathrm{Var}_{\mathbb{P}_\Pi}(S_f) = \mathrm{Var}(\Pi, f). \tag{25}$$

Definition 4.1. Let $\mathcal{V}_0(\Pi)$ be the subset of functions $f \in \mathcal{V}(\Pi)$, such that there exists an exhausting sequence of bounded subsets $(E_n)_{n\geq 1}$, depending on f, so that

$$f\chi_{E_n} \xrightarrow[n\to\infty]{\mathcal{V}(\Pi)} f.$$

The identity (25) implies that there exists a unique isometric embedding (as metric spaces)

$$\overline{S} : \mathcal{V}_0(\Pi) \to L^2(\mathrm{Conf}(E), \mathbb{P}_\Pi)$$

extending the definition (24), so that we have

$$\overline{S}_f = \lim_{n\to\infty} \sum_{x \in \mathcal{X} \cap E_n} f(x) - \mathbb{E}_{\mathbb{P}_\Pi} \sum_{x \in \mathcal{X} \cap E_n} f(x).$$

Definition 4.2. Given a non-negative function $g : E \to \mathbb{R}$ such that $\log g \in \mathcal{V}_0(\Pi)$, then we set

$$\widetilde{\Psi}_g = \exp(\overline{S}_{\log g}).$$

If moreover, $\widetilde{\Psi}_g \in L^1(\mathrm{Conf}(E), \mathbb{P}_\Pi)$, then we set

$$\overline{\Psi}_g = \frac{\widetilde{\Psi}_g}{\mathbb{E}_{\mathbb{P}_\Pi} \widetilde{\Psi}_g}.$$

The function $\overline{\Psi}_g$ is called the regularized multiplicative functional associated to g and $\mathbb{P}_\Pi$. For specifying the dependence on $\mathbb{P}_\Pi$, the notation $\overline{\Psi}_g^\Pi$ will also be used. By definition, for $\mathbb{P}_\Pi$-almost every configuration $\mathfrak{X}$, the following identity holds:

$$\log \overline{\Psi}_g^\Pi(\mathfrak{X}) = \lim_{n\to\infty} \sum_{x\in\mathfrak{X}\cap E_n} \log g(x) - \mathbb{E}_{\mathbb{P}_\Pi}\left(\sum_{x\in\mathfrak{X}\cap E_n} \log g(x)\right). \tag{26}$$

Clearly, $\overline{\Psi}_g^\Pi$ is a probability density for $\mathbb{P}_\Pi$, since $\mathbb{E}_{\mathbb{P}_\Pi}(\overline{\Psi}_g^\Pi) = 1$.

Theorem 4.1. *Let $E_0 \subset E$ be a Borel subset satisfying $\mathrm{tr}(\chi_{E_0}\Pi\chi_{E_0}) < \infty$ and such that if $\varphi \in L$ satisfies $\chi_{E\setminus E_0}\varphi = 0$, then $\varphi = 0$ identically.*

Let g be a nonnegative Borel function on E satisfying $g|_{E_0} = 0$, $g|_{E_0^c} > 0$ and such that for any $\varepsilon > 0$ the subset $E_\varepsilon = \{x \in E : |g(x) - 1| \geq \varepsilon\}$ is bounded. Assume moreover that there exists an increasing sequence of bounded subsets $(E_n)_{n\geq 1}$ exhausting the whole phase space E and

$$\int\limits_{E_n} |g(x) - 1|\Pi(x,x)d\mu(x) < \infty; \tag{27}$$

$$\int\limits_{E_n^c} |g(x) - 1|^3\Pi(x,x)d\mu(x) < \infty; \tag{28}$$

$$\iint\limits_{E_n^c\times E_n^c} |g(x) - g(y)|^2|\Pi(x,y)|^2 d\mu(x)d\mu(y) < \infty. \tag{29}$$

And

$$\lim_{n\to\infty} \mathrm{tr}(\chi_{E_n}\Pi|g-1|^2\chi_{E_n^c}\Pi\chi_{E_n}) = 0. \tag{30}$$

Then

$$\widetilde{\Psi}_g \in L^1(\mathrm{Conf}(E), \mathbb{P}_\Pi).$$

If the subspace $\sqrt{g}L$ is closed and the corresponding operator of orthogonal projection Π^g satisfies, for sufficiently large R, the condition

$$\mathrm{tr}(\chi_{g>R}\Pi^g\chi_{g>R}) < \infty \tag{31}$$

then we also have

$$\mathbb{P}_{\Pi^g} = \overline{\Psi}_g^\Pi \cdot \mathbb{P}_\Pi.$$

Remark 4.1. Note that

$$\operatorname{tr}(\chi_{E_n}\Pi|g-1|^2\chi_{E_n^c}\Pi\chi_{E_n}) = \int_{E_n} d\mu(y) \int_{E_n^c} |g(x)-1|^2|\Pi(x,y)|^2 d\mu(x).$$

Remark 4.2. The above theorem is an extension of Proposition 4.6 of [1]: we replace the convergence of $\int_{E_\varepsilon^c} |g(x)-1|^2\Pi(x,x)d\mu(x)$ in Proposition 4.6 of [1] by the convergence of $\int_{E_\varepsilon^c} |g(x)-1|^3\Pi(x,x)d\mu(x)$. This extension is crucial for treating the case of Fock space, since the former condition is already violated in the case of the Ginibre point process.

5 Case of $\mathbb{C}$

5.1 Examples

In this section, we assume that $\psi : \mathbb{C} \to \mathbb{R}$ is a measurable function on $\mathbb{C}$, the condition (1) is not necessarily satisfied. Recall that we denote $dv_\psi(z) = e^{-2\psi(z)}d\lambda(z)$ and denote $\mathscr{F}_\psi = \left\{ f : \mathbb{C} \to \mathbb{C} \middle| f \text{ holomorphic}, \int\limits_{\mathbb{C}} |f|^2 dv_\psi < \infty \right\}$. If the evaluation functionals $\mathrm{ev}_z(f) := f(z)$ defined on $\mathscr{F}_\psi$ are uniformly bounded on compact subsets, then $\mathscr{F}_\psi$ is a closed subspace of $L^2(\mathbb{C}, dv_\psi)$. In this case, denote by B_ψ the reproducing kernel of $\mathscr{F}_\psi$, we have

$$B_\psi(z,w) = \sum_{j=1}^\infty f_j(z)\overline{f_j(w)},$$

where $(f_j)_{j=1}^\infty$ is any orthonormal basis of $\mathscr{F}_\psi$.

Assumption 2. The measure dv_ψ satisfies

(1) the evaluation functionals ev_z defined on $\mathscr{F}_\psi$ are uniformly bounded on compact subsets;

(2) the polynomials are dense in $\mathscr{F}_\psi$;

(3) $\int_{\mathbb{C}} \frac{1}{1+|z|^2} B_\psi(z,z) dv_\psi(z) < \infty$.

Example 5.1 (A radial case). Let $\alpha > 0$, and set $\psi_\alpha(z) = \frac{1}{2}|z|^\alpha$, then the measure $dv_{\psi_\alpha}(z) = e^{-|z|^\alpha}d\lambda(z)$ satisfies Assumption 2 if and only if $0 < \alpha < 2$. Indeed, the first two conditions in Assumption 2 are satisfied by dv_{ψ_α} by all $\alpha > 0$. Now one can see that the third condtion is equivalent to

$$\sum_{n=1}^\infty \frac{\|z^{n-1}\|^2_{L^2(dv_\psi)}}{\|z^n\|^2_{L^2(dv_\psi)}} < \infty. \tag{32}$$

A direct computation shows that

$$\|z^n\|^2_{L^2(dv_\psi)} = \frac{2\pi}{\alpha}\Gamma\left(\frac{2n+2}{\alpha}\right) \text{ and } \frac{\|z^{n-1}\|^2_{L^2(dv_\psi)}}{\|z^n\|^2_{L^2(v_\psi)}} \sim \frac{1}{n^{2/\alpha}}. \tag{33}$$

The series (32) converges if and only if $0 < \alpha < 2$.

Remark 5.2. As shown in Example 5.1, the third condition in Assumption 2 is too strict: indeed, it fails already for the Ginibre point process (corresponding to $\psi(z) = \frac{1}{2}|z|^2$).

Let $\mathbb{P}_{B_\psi}$ be the determinantal point process induced by the operator B_ψ. For any ℓ-tuple $\mathfrak{q} = (q_1, \dots, q_\ell) \in \mathbb{C}^\ell$ of distinct points, set

$$\mathscr{F}_\psi(\mathfrak{q}) := \left\{ f \in \mathscr{F}_\psi \middle| f(q_1) = \cdots = f(q_\ell) = 0 \right\},$$

and let $B_\psi^{\mathfrak{q}}$ denote the operator of orthogonal projection onto $\mathscr{F}_\psi^{\mathfrak{q}}$. Recall that the Palm distribution $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$ of $\mathbb{P}_{B_\psi}$ conditioned at $\mathfrak{q}$ is induced by $B_\psi^{\mathfrak{q}}$, i.e.,

$$\mathbb{P}^{\mathfrak{q}}_{B_\psi} = \mathbb{P}_{B_\psi^{\mathfrak{q}}}.$$

Given a positive integer $\ell \in \mathbb{N}$, introduce the closed subspace

$$\mathscr{F}_\psi^{(\ell)} := \left\{ f \in \mathscr{F}_\psi \middle| f(0) = f'(0) = \cdots = f^{(\ell-1)}(0) = 0 \right\}. \tag{34}$$

Denote $B_\psi^{(\ell)}$ the operator of orthogonal projection onto $\mathscr{F}_\psi^{(\ell)}$. Let $\mathbb{P}^{(\ell)}_{B_\psi}$ be the determinantal point process induced by $B_\psi^{(\ell)}$.

Remark 5.3. In general, we do not have $\mathscr{F}_\psi^{(\ell)} = z^\ell \mathscr{F}_\psi$. Indeed, let $\psi(z) = \frac{1}{2}|z|^2$, we have $z\mathscr{F}_\psi \not\subset \mathscr{F}_\psi$. This can be seen from the closed graph theorem: otherwise, the operator $M_z : \mathscr{F}_\psi \to \mathscr{F}_\psi$ of multiplication by the function z is bounded, which contradicts the explicit computation (33):

$$\|M_z\|_{\mathscr{F}_\psi \to \mathscr{F}_\psi} \geq \sup_n \frac{\|z^{n+1}\|_{\mathscr{F}_\psi}}{\|z^n\|_{\mathscr{F}_\psi}} = \infty;$$

see also the related discussion after Theorem 2 in [7].

Proposition 5.1. *If ψ satisfies Assumption 2, then for any $\ell \in \mathbb{N}$ and any ℓ-tuple $\mathfrak{q} = (q_1, \dots, q_\ell) \in \mathbb{C}^\ell$ of distinct points, we have equivalence of measures:*

$$\mathbb{P}^{\mathfrak{q}}_{B_\psi} \simeq \mathbb{P}^{(\ell)}_{B_\psi}.$$

Moreover, if one sets

$$g_{\mathfrak{q}}(z) = \left| \frac{(z-q_1)\dots(z-q_\ell)}{z^\ell} \right|^2,$$

then the Radon-Nikodym derivative is given by the regularized multiplicative functional

$$\frac{d\mathbb{P}^{\mathfrak{q}}_{B_\psi}}{d\mathbb{P}^{(\ell)}_{B_\psi}} = \overline{\Psi}^{B^{(\ell)}_\psi}_{g_{\mathfrak{q}}}.$$

In particular, given any two ℓ-tuples $\mathfrak{q}$ and $\mathfrak{q}'$ of distinct points, the corresponding Palm measures $\mathbb{P}^{\mathfrak{q}}_{B_\psi}$ and $\mathbb{P}^{\mathfrak{q}'}_{B_\psi}$ are equivalent.

Proof. First note that, under Assumption 2, for any $\ell \in \mathbb{N}$ and any ℓ-tuple $\mathfrak{q} = (q_1, \dots, q_\ell) \in \mathbb{C}^\ell$ of distinct points,

$$\mathscr{F}_\psi(\mathfrak{q}) = \frac{(z-q_1)\dots(z-q_\ell)}{z^\ell}\mathscr{F}^{(\ell)}_\psi.$$

Next we use Proposition 4.6 of [1]. We now verify the assumption of Proposition 4.6 of [1] for the pair $(B^{(\ell)}_\psi, g)$. Note that $B^{(\ell)}_\psi(z,z) = O(|z|^{2\ell})$ for $|z| \to 0$ and $|g(z)-1|^2 = O\left(1/|z|^2\right)$, for $|z| \to \infty$. Recall that $B^{(\ell)}_\psi(z,z) \le B_\psi(z,z)$. Hence, under Assumption 2, we have

$$\int\limits_{|z|\le 1} |g(z)-1| B^{(\ell)}_\psi(z,z) dv_\psi(z) + \int\limits_{|z|\ge 1} |g(z)-1|^2 B^{(\ell)}_\psi(z,z) dv_\psi(z) < \infty.$$

The pair $(B^{(\ell)}_\psi, g)$ satisfies all assumptions of Proposition 4.6 in [1], and Proposition 5.1 follows immediately. □

5.2 Proof of Theorem 1.1

We now derive Theorem 1.1 from Theorem 4.1. From now on, the function ψ is assumed to satisfy the condition (1) until the end of this paper.

Let $\ell \ge 1$ and let $\mathfrak{p} = (p_1, \dots, p_\ell), \mathfrak{q} = (q_1, \dots, q_\ell) \in \mathbb{C}^\ell$ be any two fixed ℓ-tuples of distinct points; let g be the function defined by the formula

$$g(z) = |g_{\mathfrak{p},\mathfrak{q}}(z)|^2 = \left| \frac{(z-p_1)\cdots(z-p_\ell)}{(z-q_1)\cdots(z-q_\ell)} \right|^2. \tag{35}$$

Let $0 < \varepsilon < 1$ be a small fixed number. Choose $R_\varepsilon > \max\{|p_k|, |q_k| : k = 1, \dots, \ell\}$, large enough, such that outside $E_\varepsilon := \{z \in \mathbb{C} : |z| \le R_\varepsilon\}$, we have $|g(z)-1| \le \varepsilon$. Finally, for $n \in \mathbb{N}$, let $E_n = \{z \in \mathbb{C} : |z| \le \max(R_\varepsilon, n)\}$.

We start with a simple but very useful observation that conditions (28), (29), (30) and (31) in Theorem 4.1 are preserved under taking finite rank pertubation.

Remark 5.4. Assume that the pair (g, Π) satisfies the conditions (28), (29), (30) and (31) in Theorem 4.1. If $\widetilde{\Pi} = \Pi + \Pi'$, where Π' has finite rank and $\mathrm{Ran}(\Pi) \perp \mathrm{Ran}(\Pi')$, or $\widetilde{\Pi} = \Pi - \Pi'$, where Π' has finite rank and $\mathrm{Ran}(\Pi') \subset \mathrm{Ran}(\Pi)$, then conditions (28), (29), (30) and (31) hold for the new pair $(g, \widetilde{\Pi})$. If g is unbounded, then the condition (27) for the pair (g, Π) does not imply the condition for the pair $(g, \widetilde{\Pi})$. The condition (27) is on the other hand usually easy to check directly.

Lemma 5.2. *Let g be the function defined by the formula (35). We have*

$$\int_{E_n} |g(z)-1| B_\psi^{\mathfrak{q}}(z,z) e^{-2\psi(z)} d\lambda(z) < \infty \text{ and } \int_{E_n^c} |g(z)-1|^3 B_\psi^{\mathfrak{q}}(z,z) e^{-2\psi(z)} d\lambda(z) < \infty.$$

Proof. We first note that for any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that if $|z - q_k| < \varepsilon$, then

$$B_\psi^{\mathfrak{q}}(z,z) \leq C_\varepsilon |z-q_k|^2. \tag{36}$$

Indeed, $B_\psi^{\mathfrak{q}}$ is the orthogonal projection to the subspace $\mathscr{F}_\psi(\mathfrak{q})$, hence we have

$$B_\psi^{\mathfrak{q}}(z,w) = \sum_{k=1}^{\infty} f_j(z)\overline{f_j(w)}, \tag{37}$$

where $(f_j)_{j=1}^\infty$ is any orthornomal basis of $\mathscr{F}_\psi(\mathfrak{q})$. The convergence is uniform on any compact subset of $\mathbb{C}$. Thus the function $|g(z)-1| B_\psi^{\mathfrak{q}}(z,z) e^{-2\psi(z)}$ is bounded on E_n, this implies the first inequality in the lemma.

By Theorem 3.1, there exists a constant $C > 0$, such that

$$B_\psi^{\mathfrak{p}}(z,z) e^{-2\psi(z)} \leq B_\psi(z,z) e^{-2\psi(z)} \leq C.$$

Since $|g(z)-1|^3 = O(1/|z|^3)$ as $|z| \to \infty$, there exists $C' > 0$, such that

$$\int_{E_n^c} |g(z)-1|^3 B_\psi^{\mathfrak{q}}(z,z) e^{-2\psi(z)} d\lambda(z) \leq C' \int_{|z|\geq R_\varepsilon} \frac{1}{|z|^3} d\lambda(z) < \infty.$$

□

Lemma 5.3. *Let g be the function defined by the formula (35). We have*

$$\iint_{E_\varepsilon^c \times E_\varepsilon^c} |g(z)-g(w)|^2 |B_\psi^{\mathfrak{p}}(z,w)|^2 dv_\psi(z) dv_\psi(w) < \infty. \tag{38}$$

Proof. Since $B_\psi^{\mathfrak{p}}$ is a finite rank perturbation of B_ψ, and since g is bounded on E_ε^c, it suffices to show that

$$I_1 := \iint_{|z|\geq R_\varepsilon, |w| \geq R_\varepsilon} |g(z)-g(w)|^2 |B_\psi(z,w)|^2 dv_\psi(z) dv_\psi(w) < \infty. \tag{39}$$

Christ's pointwise estimate, (21) in Theorem 3.1, implies that there exists $\alpha \in \mathbb{C}$, such that

$$g(z) = 1 + \frac{\alpha}{z} + \frac{\bar{\alpha}}{\bar{z}} + O\left(1/|z|^2\right) \text{ as } |z| \to \infty.$$

Thus it suffices to show that

$$I_2 := \iint_{|z|\geq R_\varepsilon, |w|\geq R_\varepsilon} \left|\frac{1}{z} - \frac{1}{w}\right|^2 e^{-\delta|z-w|} d\lambda(z) d\lambda(w) < \infty. \tag{40}$$

To this end, write

$$\begin{aligned}
I_2 = & \int_{|w|\geq R_\varepsilon} d\lambda(w) \int_{|\zeta+w|\geq R_\varepsilon} \frac{|\zeta|^2}{|w(w+\zeta)|^2} e^{-\delta|\zeta|} d\lambda(\zeta) \\
\leq & \int_{|w|\geq R_\varepsilon} d\lambda(w) \int_{|\zeta+w|\geq R_\varepsilon} \chi_{\{|w|\geq 2|\zeta|\}} \frac{|\zeta|^2}{|w(w+\zeta)|^2} e^{-\delta|\zeta|} d\lambda(\zeta) \\
& + \int_{|w|\geq R_\varepsilon} d\lambda(w) \int_{|\zeta+w|\geq R_\varepsilon} \chi_{\{|w|< 2|\zeta|\}} \frac{|\zeta|^2}{|w(w+\zeta)|^2} e^{-\delta|\zeta|} d\lambda(\zeta).
\end{aligned}$$

The first integral is controlled by

$$4\int_{|w|\geq R_\varepsilon} d\lambda(w) \int_{\mathbb{C}} \frac{|\zeta|^2}{|w|^4} e^{-\delta|\zeta|} d\lambda(\zeta) < \infty,$$

while the second integral is controlled by

$$\begin{aligned}
& \int_{|w|\geq R_\varepsilon} d\lambda(w) \int_{\mathbb{C}} \chi_{\{|w|<2|\zeta|\}} \frac{|\zeta|^2}{|R_\varepsilon w|^2} e^{-\delta|\zeta|} d\lambda(\zeta) \\
= & 2\pi \int_{2|\zeta|\geq R_\varepsilon} \log\left(\frac{2|\zeta|}{R_\varepsilon}\right) \frac{|\zeta|^2}{|R_\varepsilon|^2} e^{-\delta|\zeta|} d\lambda(\zeta) < \infty.
\end{aligned}$$

The proof of the lemma is complete. □

Lemma 5.4. *Let g be the function defined by the formula* (35). *We have*

$$\lim_{n\to\infty} \mathrm{tr}(\chi_{E_n} B_\psi^{\mathfrak{p}} |g-1|^2 \chi_{E_n^c} B_\psi^{\mathfrak{p}} \chi_{E_n}) = 0. \tag{41}$$

Proof. Since $B_\psi^{\mathfrak{p}}$ is a finite rank perturbation of B_ψ, by Remark 5.4, it suffices to check the same condition (41) for the new pair (g, B_ψ). By applying again Christ's pointwise estimate (21), we have

$$\begin{aligned}
I_3(n) := & \mathrm{tr}(\chi_{E_n} B_\psi |g-1|^2 \chi_{E_n^c} B_\psi \chi_{E_n}) = \| \chi_{E_n} B_\psi |g-1| \chi_{E_n^c} \|_{HS}^2 \\
= & \int\limits_{|z|\leq n} \int\limits_{|w|\geq n} |g(w)-1|^2 |B_\psi(z,w)|^2 e^{-2\psi(z)-2\psi(w)} d\lambda(z) d\lambda(w) \\
\leq & C \int\limits_{|z|\leq n} \int\limits_{|w|\geq n} |g(w)-1|^2 e^{-\delta|z-w|} d\lambda(z) d\lambda(w) \\
\leq & C' \int\limits_{|z|\leq n} \int\limits_{|w|\geq n} \frac{1}{|w|^2} e^{-\delta|z-w|} d\lambda(z) d\lambda(w) = C' \int\limits_{|w|\geq n} \frac{d\lambda(w)}{|w|^2} \int\limits_{|w+\zeta|\leq n} e^{-\delta|\zeta|} d\lambda(\zeta) \\
\leq & C' \int\limits_{|w|\geq n} \frac{d\lambda(w)}{|w|^2} \int\limits_{|w|-n\leq|\zeta|\leq|w|+n} e^{-\delta|\zeta|} d\lambda(\zeta) = 4\pi^2 C' \int_{s\geq n} \frac{ds}{s} \int_{s-n}^{s+n} r e^{-\delta r} dr.
\end{aligned}$$

Now since there exists $C'' > 0$, such that $re^{-\delta r} \leq C'' e^{-\delta r/2}$ for all $r \geq 0$, we have

$$I_3(n) \leq C''' \int\limits_{s \geq n} \frac{e^{-\delta(s-n)/2}}{s} ds = C''' \int_1^\infty \frac{e^{-\delta n(t-1)/2}}{t} dt.$$

By dominated convergence theorem, we have

$$\lim_{n \to \infty} I_3(n) = 0.$$

$\square$

Proof of Theorem 1.1. By Lemma 5.2, Lemma 5.3 and Lemma 5.4, the conditions (27), (28), (29), (30) are satisfied by the pair $(g, B_\psi^{\mathfrak{q}})$. Moreover, let

$$\alpha(z) = \frac{|g_{\mathfrak{p},\mathfrak{q}}(z)|}{g_{\mathfrak{p},\mathfrak{q}}(z)},$$

then by Proposition 1.7, we have

$$\sqrt{g(z)} \mathscr{F}_\psi(\mathfrak{q}) = \alpha(z) g_{\mathfrak{p},\mathfrak{q}}(z) \mathscr{F}_\psi(\mathfrak{q}) = \alpha(z) \mathscr{F}_\psi(\mathfrak{p}).$$

Hence $\sqrt{g(z)}\mathscr{F}_\psi(\mathfrak{q})$ is a closed subspace of $L^2(dv_\psi)$. And $(B_\psi^{\mathfrak{q}})^g = \alpha B_\psi^{\mathfrak{p}} \bar{\alpha}$ is locally of trace class, this implies the condition (31). Now the formula (2) of Radon-Nikodym derivative $d\mathbb{P}_{B_\psi}^{\mathfrak{p}} / d\mathbb{P}_{B_\psi}^{\mathfrak{q}}$ follows from Theorem 4.1 and Remark 2.1. $\square$

Remark 5.5. Under the condition (1), we also have the same result as in Proposition 5.1.

5.3 Proof of Proposition 1.2

Lemma 5.5. *There exists a constant $C > 0$, depending only on ψ, such that for any C^2-smooth compactly supported function $\varphi : \mathbb{C} \to \mathbb{R}$, we have*

$$\mathrm{Var}_{\mathbb{P}_{B_\psi}}(S_\varphi) \leq C \int_{\mathbb{C}} \|\nabla \varphi(w)\|_2^2 d\lambda(w). \tag{42}$$

Proof. Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a C^2-smooth compactly supported function. Our convention for the Fourier transform of φ will be

$$\widehat{\varphi}(\xi) = \int_{\mathbb{C}} \varphi(w) e^{-i2\pi\langle w, \xi \rangle} d\lambda(w), \text{ where } \langle z, w \rangle := \Re(z)\Re(w) + \Im(z)\Im(w).$$

By definition, we have

$$\mathrm{Var}_{\mathbb{P}_{B_\psi}}(S_\varphi) = \frac{1}{2} \iint_{\mathbb{C}^2} |\varphi(z) - \varphi(w)|^2 |B_\psi(z,w)|^2 e^{-2\psi(z) - 2\psi(w)} d\lambda(z) d\lambda(w).$$

By Theorem 3.1 and Plancherel identity for Fourier transform, we obtain

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{B_\psi}}(S_\varphi) &\le C \iint_{\mathbb{C}^2} |\varphi(z)-\varphi(w)|^2 e^{-\delta|z-w|} d\lambda(z)d\lambda(w)\\ &= C \iint_{\mathbb{C}^2} |\varphi(\zeta+w)-\varphi(w)|^2 e^{-\delta|\zeta|} d\lambda(w)d\lambda(\zeta)\\ &= C \iint_{\mathbb{C}^2} |e^{i2\pi\langle\xi,\zeta\rangle}-1|^2|\widehat{\varphi}(\xi)|^2 e^{-\delta|\zeta|} d\lambda(\xi)d\lambda(\zeta).\end{aligned}$$

Now since $|e^{i2\pi\langle\xi,\zeta\rangle}-1| = 2|\sin(\pi\langle\xi,\zeta\rangle)| \le 2\pi|\xi||\zeta|$, we have

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{B_\psi}}(S_\varphi) &\le C' \iint_{\mathbb{C}^2} |\xi|^2|\widehat{\varphi}(\xi)|^2|\zeta|^2 e^{-\delta|\zeta|} d\lambda(\xi)d\lambda(\zeta)\\ &\le C'' \int_{\mathbb{C}} |\xi|^2|\widehat{\varphi}(\xi)|^2 d\lambda(\xi) = C'' \int_{\mathbb{C}} \|\nabla\varphi(w)\|_2^2 d\lambda(w).\end{aligned}$$

□

Proof of Proposition 1.2. We will follow the argument of Ghosh and Peres [9]. By Theorem 2.2, it suffices, for any fixed bounded open set $\mathcal{D}$ with Lebesgue-negligible boundary and any $\varepsilon > 0$, to construct a function $\Phi_\varepsilon \in C_c^2(\mathbb{C})$ such that $\Phi_\varepsilon|_{\mathcal{D}} \equiv 1$ and $\mathrm{Var}_{\mathbb{P}_{B_\psi}}(S_{\Phi_\varepsilon}) < \varepsilon$.

Let $r_0 = 2\sup\{|z| : z \in \mathcal{D}\}$. By Lemma 5.5, it suffices to construct a radial function

$$\Phi_\varepsilon(z) = \phi_\varepsilon(|z|),$$

with ϕ_ε a function in $C_c^2(\mathbb{R}_+)$ such that $\phi_\varepsilon|_{[0,r_0/2]} \equiv 1$ and

$$\int_0^\infty |\phi_\varepsilon'(r)|^2 r dr < \varepsilon.$$

To this end, first we take $\tilde{\phi}_\varepsilon(r) = (1-\varepsilon\log^+(r/r_0))_+$, where $\log^+(x) = \max(\log x, 0)$. Note that $\tilde{\phi}_\varepsilon|_{[r_0\exp(1/\varepsilon),\infty)} \equiv 0$ and $\tilde{\phi}_\varepsilon'(r) = -\varepsilon/r$ on the interval $(r_0, r_0\exp(1/\varepsilon))$. Next we smooth the function $\tilde{\phi}_\varepsilon$ at the points r_0 and $r_0\exp(1/\varepsilon)$ to obtain a function $\phi_\varepsilon \in C_c^2(\mathbb{R}_+)$ such that ϕ_ε identically equals to 1 on $[0, r_0/2]$ and ϕ_ε' is supported on $[r_0/2, 2r_0\exp(1/\varepsilon)]$ such that $|\phi_\varepsilon'(r)| \le \varepsilon/r$ for all $r > 0$. Hence we have

$$\int_0^\infty |\phi_\varepsilon'(r)|^2 r dr \le \int_{r_0/2}^{2r_0\exp(1/\varepsilon)} \frac{\varepsilon^2}{r} dr = \varepsilon + \varepsilon^2\log 4.$$

This completes the proof of the proposition. □

6 Case of $\mathbb{D}$

6.1 Analysis of the conditions on the weight ω

Let $\omega : \mathbb{D} \to \mathbb{R}^+$ be a Bergman weight. We collect some known results from the literature on the sufficient conditions on the Bergman weight ω, so that the inequality (3):

$$\int_{\mathbb{D}} (1-|z|)^2 B_\omega(z,z)\omega(z) d\lambda(z) < \infty$$

holds.

Example 6.1 (Classical weights). Assume $\omega(z) = (1-|z|^2)^\alpha, \alpha > -1$. Then

$$B_\omega(z,w) = \frac{\alpha+1}{\pi} \frac{1}{(1-z\bar{w})^{\alpha+2}},$$

hence $(1-|z|)^2 B_\omega(z,z)\omega(z)$ is bounded and the inequality (3) holds.

Example 6.2 (A class of logarithmatically superharmonic weights). Let

$$\omega(z) = e^{-2\varphi(z)}.$$

Assume

1) $\varphi \in C^2(\mathbb{D})$ and $\Delta\varphi > 0$;

2) the function $(\Delta\varphi(z))^{-1/2}$ is Lipschitz on $\mathbb{D}$;

3) there exist $C_1, a > 0$ and $0 < t < 1$, such that

$$(\Delta\varphi(z))^{-1/2} \le C_1(1-|z|);$$

$$(\Delta\varphi(z))^{-1/2} \le (\Delta\varphi(w))^{-1/2} + t|z-w| \ \text{ for } \ |z-w| > a(\Delta\varphi(w))^{-1/2}.$$

By [13, Lemma 3.5], the weight ω is a Bergman weight and

$$\sup_{z\in\mathbb{D}} (1-|z|)^2 B_\omega(z,z)\omega(z) < \infty.$$

Hence the inequality (3) holds. Some concrete such examples are

- $\omega(z) = (1-|z|^2)^\alpha \exp(h(z))$ with $\alpha > 0$ and $h(z)$ any real harmonic function on $\mathbb{D}$;
- $\omega(z) = (1-|z|^2)^\alpha \exp(-\beta(1-|z|^2)^{-\gamma} + h(z))$ with $\alpha \ge 0, \beta > 0, \gamma > 0$ and $h(z)$ any real harmonic function on $\mathbb{D}$.

Proposition 6.1. *Let ω_1, ω_2 be two Bergman weights on $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} (1-|z|)^2 B_{\omega_1}(z,z)\omega_2(z)d\lambda(z) < \infty.$$

Let ω be a Bergman weight on $\mathbb{D}$ and assume that there exist $c, C > 0$ such that

$$c\omega_1(z) \leq \omega(z) \leq C\omega_2(z)$$

then ω satisfies the condition (3).

Proof. Since $B_\omega(z,z) = \sup_{\|f\|_{\mathscr{B}_\omega \leq 1}} |f(z)|^2$, we have $B_\omega(z,z) \leq c^2 B_{\omega_1}(z,z)$. By the assumption, we have

$$\int_{\mathbb{D}} (1-|z|)^2 B_\omega(z,z)\omega(z)d\lambda(z) \leq c^2 C \int_{\mathbb{D}} (1-|z|)^2 B_{\omega_1}(z,z)\omega_2(z)d\lambda(z) < \infty.$$

□

Example 6.3. Let ω be a Bergman weight. Assume that there exist $c, C > 0$ and let α, β be either $0 \geq \alpha \geq \beta > -1$ or $\alpha \geq \beta > \alpha - 1 \geq -1$, such that

$$c(1-|z|^2)^\alpha \leq \omega(z) \leq C(1-|z|^2)^\beta$$

then ω satisfies the condition (3).

6.2 Proof of Theorem 1.4 and Proposition 1.5

Let $k, \ell \in \mathbb{N} \cup \{0\}$, let $\mathfrak{p} \in \mathbb{D}^\ell$ be an ℓ-tuple of *distinct* points and $\mathfrak{q} \in \mathbb{D}^k$ a k-tuple of distinct points. Set

$$g(z) = |b_{\mathfrak{p}}(z) b_{\mathfrak{q}}(z)^{-1}|^2 = \prod_{j=1}^{\ell} \left| \frac{z - p_j}{1 - \bar{p}_j z} \right|^2 \cdot \prod_{j=1}^{k} \left| \frac{1 - \bar{q}_j z}{z - q_j} \right|^2.$$

By virtue of Proposition 1.8, to prove Proposition 1.5 and hence Theorem 1.4, it suffices to prove that the pair $(g, B_\omega^{\mathfrak{q}})$ satisfies the assumption of Proposition 4.6 of [1]. This is done in the following

Lemma 6.2. *Take $\varepsilon > 0$ small enough and let $E_\varepsilon = \bigcup_{i=1}^k U_\varepsilon(q_i)$, where $U_\varepsilon(q_i)$ is a disc centred at point q_i with radius ε in $\mathbb{D}$. Then we have*

$$\int_{E_\varepsilon} |g(z) - 1| B_\omega^{\mathfrak{q}}(z,z)\omega(z)d\lambda(z) + \int_{E_\varepsilon^c} |g(z) - 1|^2 B_\omega^{\mathfrak{q}}(z,z)\omega(z)d\lambda(z) < \infty. \quad (43)$$

Proof. For $\varepsilon > 0$ small enough, there exists $C > 0$ such that for any $z \in E_\varepsilon$, we have

$$B_\omega^{\mathfrak{q}}(z,z) \le C \prod_{i=1}^{k} |z - q_i|^2,$$

whence $|g(z) - 1| B_\omega^{\mathfrak{q}}(z,z)$ is bounded on E_ε, and the first integral in (43) is bounded.

For the second integral, the identities

$$\left| \frac{z - p_j}{1 - \bar{p}_j z} \right|^2 = 1 - \frac{(1 - |z|^2)(1 - |p_j|^2)}{|1 - \bar{p}_j z|^2},$$

together with the same identities for $q_j : j = 1, \dots, k$, imply that there exists $C' > 0$ such that

$$|g(z) - 1| \le C'(1 - |z|) \text{ for } z \in E_\varepsilon^c.$$

Note also that since $\mathrm{Ran}(B_\omega^{\mathfrak{q}}) \subset \mathrm{Ran}(B_\omega)$, we have $B_\omega^{\mathfrak{q}}(z,z) \le B_\omega(z,z)$, hence by our assumption (3), we have

$$\int_{E_\varepsilon^c} |g(z) - 1|^2 B_\omega^{\mathfrak{q}}(z,z) \omega(z) d\lambda(z) \le C' \int_{E_\varepsilon^c} (1 - |z|)^2 B_\omega(z,z) \omega(z) d\lambda(z) < \infty.$$

$\square$

7 Proof of Theorem 4.1

Recall that we denote by Π an orthogonal projection on $L^2(E, \mu)$ which is locally in trace class.

In [1], a class of Borel functions on E, denoted there by $\mathscr{A}_2(\Pi)$, plays a central role in the proof of the main result. Recall that, by definition, $\mathscr{A}_2(\Pi)$ is the set of positive Borel functions g on E satisfying

(1) $0 < \inf_E g \le \sup_E g < \infty$;

(2) $\int_E |g(x) - 1|^2 \Pi(x,x) d\mu(x) < \infty$.

If $g \in \mathscr{A}_2(\Pi)$, then the subspace $\sqrt{g} L$, where L is the range of the orthogonal projection Π, is automatically closed; we set Π^g to be the corresponding operator of orthogonal projection. The main property of $\mathscr{A}_2(\Pi)$ that will be used later is stated in the following

Proposition 7.1 (Cor. 4.11 of [1]). *If $g \in \mathscr{A}_2(\Pi)$ satisfies*

$$\sup_E |g(x) - 1| < 1.$$

Then the operator Π^g is locally of trace class, and we have

$$\mathbb{P}_{\Pi^g} = \overline{\Psi}_g^{\Pi} \cdot \mathbb{P}_{\Pi}. \tag{44}$$

Let $g : E \to \mathbb{R}$ be a Borel function, set

$$L(g) := \int_E |g(x) - 1|^3 \Pi(x, x) d\mu(x) \in [0, \infty] \tag{45}$$

and

$$V(g) := \iint_{E^2} |g(x) - g(y)|^2 |\Pi(x, y)|^2 d\mu(x) d\mu(y) \in [0, \infty]. \tag{46}$$

And then, we introduce a new class of Borel functions on E as follows. Let $\mathscr{A}_3(\Pi)$ be the set of positive Borel functions g on E satisfying

(1) $0 < \inf_E g \leq \sup_E g < \infty$;

(2) $L(g) = \int_E |g(x) - 1|^3 \Pi(x, x) d\mu(x) < \infty$;

(3) $V(g) = \iint_{E^2} |g(x) - g(y)|^2 |\Pi(x, y)|^2 d\mu(x) d\mu(y) < \infty$;

(4) there exists an exhausting sequence $(E_n)_{n \geq 1}$ of bounded subsets of E, possibly depending on g, such that

$$\lim_{n \to \infty} \mathrm{tr}(\chi_{E_n} \Pi |g - 1|^2 \chi_{E_n^c} \Pi \chi_{E_n}) = 0. \tag{47}$$

More precisely, Relation (47) can equivalently be rewritten as follows:

$$\lim_{n \to \infty} \iint_{E^2} \chi_{E_n^c}(x) \chi_{E_n}(y) |g(x) - 1|^2 |\Pi(x, y)|^2 d\mu(x) d\mu(y) = 0. \tag{48}$$

Remark 7.1. We have the following useful identity

$$V(g) = \|[g, \Pi]\|_{HS}^2, \tag{49}$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm and $[g, \Pi] = g\Pi - \Pi g$ is the commutator of the operator of multiplication by g and the projection operator Π.

Remark 7.2. The sequence $(E_n)_{n \geq 1}$ in the definition of $\mathscr{A}_3(\Pi)$ is an analogue of the sequence of the subsets $(\{z \in \mathbb{C} : |z| \leq n\})_{n \geq 1}$ in the proof of Lemma 5.4.

The most technical result in this section is the following

Proposition 7.2. *If $g \in \mathscr{A}_3(\Pi)$ satisfies*

$$\sup_E |g(x) - 1| < 1. \tag{50}$$

Then the operator Π^g is locally of trace class, and we have

$$\mathbb{P}_{\Pi^g} = \overline{\Psi}_g^{\Pi} \cdot \mathbb{P}_{\Pi}. \tag{51}$$

Remark 7.3. Note that the condition (47) holds automatically for any $g \in \mathscr{A}_2(\Pi)$, hence we have

$$\mathscr{A}_2(\Pi) \subset \mathscr{A}_3(\Pi).$$

Proof of Theorem 4.1. We now derive Theorem 4.1 from Proposition 7.2. The proof is similar to the proof of Proposition 4.6 of [1]. Proving the statement for $\mathscr{A}_3(\Pi)$ instead of $\mathscr{A}_2(\Pi)$ requires extra effort, however. For sake of completeness, let us sketch the proof here.

Let $\mathrm{Conf}(E; E \setminus E_0))$ stand for the subset of $\mathrm{Conf}(E)$ consisting of those configurations whose particles all lie in $E \setminus E_0$. The assumptions of Theorem 4.1 imply that $\mathbb{P}_\Pi(\mathrm{Conf}(E; E \setminus E_0)) > 0$. Replacing, if necessary, g by $g|_{E_0^c}$ and L by $\chi_{E_0^c} L$, we may assume that g is positive on E.

By our assumption, we may choose $0 < \varepsilon_1 < \varepsilon_2 < 1$ and a bounded subset $E_1 \subset E$, such that

$$\{x \in E : |g(x) - 1| \geq \varepsilon_2\} \subset E_1 \subset \{x \in E : |g(x) - 1| \geq \varepsilon_1\},$$

and

$$\|\chi_{\{x \in E: |g(x)-1| \leq \varepsilon_2\}} \Pi\| < 1.$$

Decompose $E_1 = E_1^+ \sqcup E_1^-$ by setting

$$E_1^+ = \{x \in E : g(x) > 1\} \cap E_1 \text{ and } E_1^- = \{x \in E : g(x) < 1\} \cap E_1.$$

Note that

$$E_1^+ \subset \{x \in E : g(x) > 1 + \varepsilon_1\} \text{ and } E_1^- \subset \{x \in E : g(x) < 1 - \varepsilon_1\}.$$

Then we can decompose g as $g = g_1 g_2 g_3$ with

$$\begin{aligned} g_1 &= (g-1)\chi_{E_1^c} + 1, \\ g_2 &= (g-1)\chi_{E_1^-} + 1, \\ g_3 &= (g-1)\chi_{E_1^+} + 1. \end{aligned}$$

Claim. We have $g_1 \in \mathscr{A}_3(\Pi)$.

Indeed, the first two and the last condition in the definition of $\mathscr{A}_3(\Pi)$ are immediate for g_1. We now check the third condition. We have

$$|g_1(x) - g_1(y)| = \begin{cases} |g(x) - g(y)| & (x, y) \in E_1^c \times E_1^c \\ |g(x) - 1| & (x, y) \in E_1^c \times E_1 \\ |g(y) - 1| & (x, y) \in E_1 \times E_1^c \\ 0 & (x, y) \times E_1 \times E_1 \end{cases},$$

whence

$$\begin{aligned}V(g_1) &= \iint_{E^2} |g_1(x)-g_1(y)|^2 |\Pi(x,y)|^2 d\mu(x)d\mu(y)\\ &= \iint_{E_1^c\times E_1^c} |g(x)-g(y)|^2 |\Pi(x,y)|^2 d\mu(x)d\mu(y)\\ &\quad + 2\int_{E_1} d\mu(y)\int_{E_1^c} |g(x)-1|^2 |\Pi(x,y)|^2 d\mu(x).\end{aligned}$$

By (29), (30) and Remark 4.1, we have $V(g_1)<\infty$.

By Proposition 7.2, we have

$$\mathbb{P}_{\Pi^{g_1}} = \overline{\Psi}_{g_1}^{\Pi}\cdot \mathbb{P}_{\Pi}.$$

The rest of the proof of Theorem 4.1 follows the scheme of the proof of Proposition 4.6 of [1]. First, we have

$$\Pi^{g_1g_2} = (\Pi^{g_1})^{g_2} \text{ and } \Pi^{g} = \Pi^{g_1g_2g_3} = (\Pi^{g_1g_2})^{g_3}.$$

Since g_2 is bounded and g_2-1 is compactly supported, the usual multiplicative functional

$$\Psi_{g_2}(\mathcal{X}) = \prod_{x\in\mathcal{X}} g_2(x),$$

is well defined and

$$\mathbb{P}_{\Pi^{g_1g_2}} = C_1\Psi_{g_2}\mathbb{P}_{\Pi^{g_1}}.$$

The function g_3-1, although not necessarily bounded, is compactly supported and positive. The usual multiplicative functional Ψ_{g_3} is also well defined for $\mathbb{P}_{\Pi^{g_1g_2}}$-almost every configuration. Indeed, since g_1g_2 is bounded and by Proposition 4.1 of [1], there exists $C>0$ such that

$$\Pi^{g_1g_2}(x,x)\le C\Pi(x,x).$$

Consequently, we have

$$\int_E |g_3(x)-1|\Pi^{g_1g_2}(x,x)d\mu(x) \le C\int_{E_1^+} |g_3(x)-1|\Pi(x,x)d\mu(x) < \infty. \tag{52}$$

In the relation (52), we used the fact that g_3-1 is supported on E_1^+ and our assumption (27). It follows that

$$\mathbb{E}_{\mathbb{P}_{\Pi^{g_1g_2}}}(\Psi_{g_3}) = \det(1+(g_3-1)\Pi^{g_1g_2})<\infty.$$

Hence, by Proposition 4.4 in [1], we have

$$\mathbb{P}_{\Pi^{g}} = C'\Psi_{g_3}\mathbb{P}_{\Pi^{g_1g_2}} = C'C\Psi_{g_3}\Psi_{g_2}\cdot\mathbb{P}_{\Pi^{g_1}} = C'C\Psi_{g_3}\Psi_{g_2}\overline{\Psi}_{g_1}^{\Pi}\cdot\mathbb{P}_{\Pi},$$

whence $\mathbb{P}_{\Pi^{g}} = \overline{\Psi}_{g}^{\Pi}\mathbb{P}_{\Pi}$ and Theorem 4.1 is completely proved. $\square$

Introduce a topology $\mathscr{T}$ on $\mathscr{A}_3(\Pi)$ generated by the open sets

$$U(\varepsilon, g) = \{g' \in \mathscr{A}_N(\Pi) : L(g'/g) < \varepsilon, V(g'/g) < \varepsilon\},$$

where L, V are defined by formulae (45), (46). With respect to the topology $\mathscr{T}$, a sequence g_n tends to g in $\mathscr{A}_3(\Pi)$ if and only if

$$L(g_n/g) \to 0 \text{ and } V(g_n/g) \to 0. \tag{53}$$

Lemma 7.3. *Let $g \in \mathscr{A}_3(\Pi)$ and let $(E_n)_{n\geq 1}$ be the exhausting sequence of bounded subsets of E such that condition (47) holds. Denote*

$$g_n = 1 + (g-1)\chi_{E_n}.$$

Then

$$g_n \xrightarrow[n\to\infty]{\mathscr{T}} g.$$

Proof. Assume that $g \in \mathscr{A}_3(\Pi)$. First, by definition, we have

$$|g_n/g - 1| = |1/g - 1|\chi_{E_n^c} \leq \frac{1}{\inf_E g}|g-1|.$$

It follows that $L(g_n/g) \to 0$.

Next, setting

$$V_n(x,y) = |g_n(x)/g(x) - g_n(y)/g(y)|^2 |\Pi(x,y)|^2,$$

we have

$$V(g_n/g) = \iint\limits_{E_n \times E_n^c} V_n + \iint\limits_{E_n^c \times E_n} V_n + \iint\limits_{E_n^c \times E_n^c} V_n. \tag{54}$$

The first and second terms in (54) are equal and

$$\begin{aligned}
\iint\limits_{E_n \times E_n^c} V_n &= \iint\limits_{E_n \times E_n^c} |1 - 1/g(y)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y) \\
&\leq \frac{1}{\inf_E g^2} \iint\limits_{E_n \times E_n^c} |g(y) - 1|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y) \\
&= \frac{1}{\inf_E g^2} \|\chi_{E_n} \Pi |g-1| \chi_{E_n^c}\|_2^2 \to 0.
\end{aligned}$$

The third term in (54) converges to 0 since

$$\iint\limits_{E_n^c \times E_n^c} V_n \leq \frac{1}{\inf_E g^2} \iint\limits_{E_n^c \times E_n^c} |g(x) - g(y)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y),$$

and the latter integral tends to 0 as $n \to \infty$. Thus $V(g_n/g) \to 0$, and Lemma 7.3 is completely proved. $\square$

Lemma 7.4. *Let $g_n \in \mathscr{A}_3(\Pi), n \geq 1$, $g \in \mathscr{A}_3(\Pi)$, and assume that the sequence (g_n) is uniformly bounded. If $g_n \xrightarrow[n\to\infty]{\mathscr{T}} g$, then $L(g_n) \to L(g)$ and $V(g_n) \to V(g)$.*

Proof. By definition, we have $L(g_n/g) \to 0$ and $V(g_n/g) \to 0$.

The relation $L(g_n/g) \to 0$ together with the inequality

$$\int |g_n(x) - g(x)|^3 \Pi(x,x) d\mu(x) \leq \sup_E g \cdot \int |g_n(x)/g(x) - 1|^3 \Pi(x,x) d\mu(x)$$

implies that

$$\lim_{n\to\infty} \|(g_n - 1) - (g-1)\|_{L^3(E;\Pi(x,x)d\mu(x))} = 0,$$

whence

$$\lim_{n\to\infty} \|g_n - 1\|_{L^3(E;\Pi(x,x)d\mu(x)} = \|g-1\|_{L^3(E;\Pi(x,x)d\mu(x))}.$$

This is equivalent to $L(g_n) \to L(g)$ as $n \to \infty$.

We turn to the proof of the convergence $V(g_n) \to V(g)$. It suffices to prove any convergent subsequence (in $[0,\infty]$) of the sequence $(V(g_n))_{n\geq 1}$ converges to $V(g)$. We have already shown that

$$\int_E |g_n(x) - g(x)|^3 \Pi(x,x) d\mu(x) \to 0.$$

Passing perhaps to a subsequence, we may assume that $g_n \to g$ almost everywhere with respect to $\Pi(x,x)d\mu(x)$. Set

$$F_n(x,y) = g_n(x) - g_n(y) \text{ and } F(x,y) = g(x) - g(y).$$

The desired relation $V(g_n) \to V(g)$ is equivalent to the relation

$$\lim_{n\to\infty} \|F_n\|_{L^2(E\times E;\,|\Pi(x,y)|^2 d\mu(x)d\mu(y))} = \|F\|_{L^2(E\times E;\,|\Pi(x,y)|^2 d\mu(x)d\mu(y))}$$

To simplify notation, we denote $dM_2(x,y) = |\Pi(x,y)|^2 d\mu(x)d\mu(y)$. It suffices to prove that

$$\lim_{n\to\infty} \|F_n - F\|_{L^2(E\times E;\, dM_2)} = 0. \tag{55}$$

A direct computation shows that

$$\frac{F_n(x,y) - F(x,y)}{g(x)} = \frac{g_n(x)}{g(x)} - \frac{g_n(y)}{g(y)} + \frac{F(x,y)(g_n(y) - g(y))}{g(x)g(y)}.$$

Hence we have

$$|F_n(x,y) - F(x,y)| \leq \sup_E g \cdot \left| \frac{g_n(x)}{g(x)} - \frac{g_n(y)}{g(y)} \right| + \frac{1}{\inf_E g} |F(x,y)| \cdot |g_n(y) - g(y)|,$$

and

$$\|F_n - F\|_{L^2(E\times E;\,dM_2)} \leq \sup_E g \cdot \left\| \frac{g_n(x)}{g(x)} - \frac{g_n(y)}{g(y)} \right\|_{L^2(E\times E;\,dM_2)} + \frac{1}{\inf_E g} \|F(x,y)\cdot |g_n(y) - g(y)|\|_{L^2(E\times E;\,dM_2)}$$

The limit relation $V(g_n/g) \to 0$ implies that

$$\lim_{n\to\infty} \left\| \frac{g_n(x)}{g(x)} - \frac{g_n(y)}{g(y)} \right\|_{L^2(E\times E;\,dM_2)} = 0.$$

By definition, $F \in L^2(E \times E;\, dM_2)$. Since the sequence (g_n) is uniformly bounded and $g_n \to g$ almost everywhere with respect to $\Pi(x,x)d\mu(x)$, the dominated convergence theorem yields

$$\lim_{n\to\infty} \|F(x,y)\cdot |g_n(y) - g(y)|\|_{L^2(E\times E;\,dM_2)} = 0.$$

This completes the proof of (55). Lemma 7.4 is proved completely.

□

Recall that, in Definition 4.1 and Definition 4.2, we introduced the subset $\mathcal{V}_0(\Pi) \subset \mathcal{V}(\Pi)$ and the functional $\widetilde{\Psi}_g$ for functions g such that $\log g \in \mathcal{V}_0(\Pi)$. Recall also that we introduced in (23) the notation $\mathrm{Var}(\Pi, f)$ for any Borel function $f : E \to \mathbb{C}$.

Lemma 7.5. *If $g \in \mathscr{A}_3(\Pi)$, then*

$$\mathrm{Var}(\Pi, \log g) < \infty \textit{ and } \log g \in \mathcal{V}_0(\Pi).$$

In particular, for any function $g \in \mathscr{A}_3(\Pi)$, the functional $\widetilde{\Psi}_g$ is well-defined.

Proof. By the third condition in the definition of $\mathscr{A}_3(\Pi)$, if $g \in \mathscr{A}_3(\Pi)$, then

$$\mathrm{Var}(\Pi, g-1) < \infty.$$

Define a function

$$F(t) := \begin{cases} \frac{\log(1+t)-t}{t^2} & \text{if } t \neq 0 \\ -\frac{1}{2} & \text{if } t = 0 \end{cases},$$

so that F is continuous on $(-1,\infty)$. It follows that for any $0 < \varepsilon \leq 1$ and $M \geq 1$, there exists $C_{\varepsilon,M} > 0$, such that if $t \in [-1+\varepsilon, -1+M]$, then

$$|\log(1+t) - t| \leq C_{\varepsilon,M} t^2. \tag{56}$$

By the first condition in the definition of $\mathscr{A}_3(\Pi)$, we can apply the above inequality to $g-1$. A simple computation yields

$$|\log g(x) - \log g(y)|^2 \leq 20M^2|g(x) - g(y)|^2 + 8MC_{\varepsilon,M}^2(|g(x)-1|^3 + |g(y)-1|^3), \tag{57}$$

where $\varepsilon = \min(1, \inf_E g)$ and $M = \max(1, \sup_E g)$. Inequality (57), combined with the reproducing property

$$\Pi(x,x) = \int_E |\Pi(x,y)|^2 d\mu(y)$$

and the second and third conditions on g in the definition of $\mathscr{A}_3(\Pi)$, yields the desired result:

$$\mathrm{Var}(\Pi, \log g) < \infty.$$

We turn to the proof of the relation $\log g \in \mathcal{V}_0(\Pi)$. By definition, there exists a sequence (E_n) of exhausting bounded subsets of E, such that the relation (48) holds. It suffices to show that

$$\lim_{n\to\infty} \|\chi_{E_n} \log g - \log g\|_{\mathcal{V}(\Pi)} = \lim_{n\to\infty} \|\chi_{E_n^c} \log g\|_{\mathcal{V}(\Pi)} = 0. \tag{58}$$

We have

$$\begin{aligned}\|\chi_{E_n^c} \log g\|^2_{\mathcal{V}(\Pi)} =& \frac{1}{2} \iint_{E_n^c \times E_n^c} |\log g(x) - \log g(y)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y) \\ &+ \frac{1}{2} \iint_{E^2} \chi_{E_n^c}(x) \chi_{E_n}(y) |\log g(x)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y) \\ &+ \frac{1}{2} \iint_{E^2} \chi_{E_n^c}(y) \chi_{E_n}(x) |\log g(y)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y).\end{aligned}$$

The fact that first integral in the above identity tends to 0 when n tends to infinity follows from the fact that $\mathrm{Var}(\Pi, \log g) < \infty$. The second and the third integrals are equal, and since $\varepsilon \leq g \leq M$, we may use $|\log g(x)| \leq C_{\varepsilon,M} |g(x) - 1|$ and we get

$$\begin{aligned}&\iint_{E^2} \chi_{E_n^c}(x) \chi_{E_n}(y) |\log g(x)|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y) \\ \leq & C^2_{\varepsilon,M} \iint_{E^2} \chi_{E_n^c}(x) \chi_{E_n}(y) |g(x) - 1|^2 |\Pi(x,y)|^2 d\mu(x) d\mu(y).\end{aligned} \tag{59}$$

The assumption (48) implies that the last integral in (59) tends to 0 as n tends to infinity. This completes the proof of the desired relation (58). □

Proposition 7.6. *For any $\varepsilon, M : 0 < \varepsilon \leq 1$, $M \geq 1$, there exists a constant $C_{\varepsilon,M} > 0$ such that if $g \in \mathscr{A}_3(\Pi)$ satisfies*

$$\varepsilon \leq \inf_E g \leq \sup_E g \leq M \tag{60}$$

then

$$\log \mathbb{E} |\widetilde{\Psi}_g|^2 \leq C_{\varepsilon,M} (L(g) + V(g)). \tag{61}$$

Definition 7.1. Let $\mathscr{A}_3^{\varepsilon,M}(\Pi) \subset \mathscr{A}_3(\Pi)$ be the subset of functions satisfying the condition (60).

By definition $|\widetilde{\Psi}_g|^2 = \widetilde{\Psi}_{g^2}$. If $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)$, then

$$L(g^2) \leq 8M^3 L(g) \text{ and } V(g^2) \leq 4M^2 V(g). \tag{62}$$

Consequently, in order to establish (61), it suffices to obtain the estimate (63) in Lemma 7.7 below.

Lemma 7.7. *For any $\varepsilon, M : 0 < \varepsilon \leq 1$, $M \geq 1$, there exists a constant $C_{\varepsilon,M} > 0$ such that if $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)$, then*

$$\log \mathbb{E}\widetilde{\Psi}_g \leq C(L(g) + V(g)). \tag{63}$$

Denote

$$g^+ = 1 + \chi_{g\geq 1}(g-1) \text{ and } g^- = 1 + \chi_{g\leq 1}(g-1).$$

Then

$$g = g^+ g^- \text{ and } g^+ \geq 1, g^- \leq 1. \tag{64}$$

Our aim here is to reduce Lemma 7.7 for g to the same statement for g^+, g^-.

Lemma 7.8. *Both g^+ and g^- are in the class $\mathscr{A}_3^{\varepsilon,M}(\Pi)$, moreover, we have*

$$L(g^\pm) \leq L(g) \textit{ and } V(g^\pm) \leq V(g). \tag{65}$$

Proof. Inequalities (65) follow from the elementary inequalities

$$|g^\pm - 1| \leq |g-1| \text{ and } |g^\pm(x) - g^\pm(y)| \leq |g(x) - g(y)|. \tag{66}$$

Let $(E_n)_{n\geq 1}$ be the exhausting sequence of bounded subsets such that (47) holds. The first inequality in (66) yields the following inequalities for self-adjoint operators:

$$\chi_{E_n}\Pi|g^\pm - 1|^2 \chi_{E_n^c}\Pi\chi_{E_n} \leq \chi_{E_n}\Pi|g-1|^2\chi_{E_n^c}\Pi\chi_{E_n}.$$

Hence (47) holds for $g^\pm$ with respect to the sequence $(E_n)_{n\geq 1}$. $\square$

Denote by $\mathscr{A}_3^{\varepsilon,M}(\Pi)^+$ the subclass of functions in $\mathscr{A}_3^{\varepsilon,M}(\Pi)$ such that

$$g \in \mathscr{A}_3(\Pi) \text{ and } g \geq 1.$$

Similarly, denote by $\mathscr{A}_3^{\varepsilon,M}(\Pi)^-$ the subclass of functions in $\mathscr{A}_3^{\varepsilon,M}(\Pi)$ such that

$$g \in \mathscr{A}_3^{\varepsilon,M}(\Pi) \text{ and } g \leq 1.$$

Let

$$\mathscr{A}_3^{\varepsilon,M}(\Pi)^\pm = \mathscr{A}_3^{\varepsilon,M}(\Pi)^+ \cup \mathscr{A}_3^{\varepsilon,M}(\Pi)^-.$$

We reduce the statement of Lemma 7.7 for general g in $\mathscr{A}_3^{\varepsilon,M}(\Pi)$ to the particular case g in $\mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$. Indeed, assume that we have established (63) in the case of $\mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$, then by multiplicativity, for general g in $\mathscr{A}_3^{\varepsilon,M}(\Pi)$, we have

$$\begin{aligned}\mathbb{E}\widetilde{\Psi}_g &= \mathbb{E}(\widetilde{\Psi}_{g^+}\widetilde{\Psi}_{g^-}) \le (\mathbb{E}\widetilde{\Psi}_{g^+}^2 \cdot \mathbb{E}\widetilde{\Psi}_{g^-}^2)^{1/2} = (\mathbb{E}\widetilde{\Psi}_{(g^+)^2} \cdot \mathbb{E}\widetilde{\Psi}_{(g^-)^2})^{1/2}\\ &\le \frac{1}{2}(\mathbb{E}\widetilde{\Psi}_{(g^+)^2} + \mathbb{E}\widetilde{\Psi}_{(g^-)^2}).\end{aligned}$$

Now we may apply (63) for functions $(g^+)^2 \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^+$ and $(g^-)^2 \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^-$ respectively and use the relations (62) together with Lemma 7.8 , to obtain that

$$\begin{aligned}\mathbb{E}\widetilde{\Psi}_g &\le C'\Big[L((g^+)^2) + V((g^+)^2) + L((g^-)^2) + V((g^-)^2\Big]\\ &\le C''\Big[L(g^+) + V(g^+) + L(g^-) + V(g^-))\Big]\\ &\le C'''(L(g) + V(g)).\end{aligned}$$

We now proceed to the proof of (63) for functions g in $\mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$ and, consequently, Lemma 7.7. By definition, if $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$, then the sequences $(g_n)_{n\ge 1}$ defined in the proof of Lemma 7.3 all stay in the set $\mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$. Since

$$\|\overline{S}_{\log g_n} - \overline{S}_{\log g}\|_2^2 = \mathrm{Var}(\Pi, \log g_n/g),$$

passing perhaps to a subsequence, we may assume that

$$\widetilde{\Psi}_{g_n} = \exp(\overline{S}_{\log g_n}) \xrightarrow[n\to\infty]{a.e.} \widetilde{\Psi}_g = \exp(\overline{S}_{\log g}).$$

By Fatou's Lemma and Lemma 7.4 , it suffices to establish (63) for a function $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$ such that the subset $\{x \in E : g(x) \neq 1\}$ is bounded. We will assume the boundedness of $\{x \in E : g(x) \neq 1\}$ until the end of the proof of Proposition 7.6.

For any $0 < \varepsilon \le 1$ and any $M \ge 1$, there exists $C_{\varepsilon,M} > 0$ such that if $t \in [-1+\varepsilon, -1+M]$, then

$$\left|\log(1+t) - t + \frac{1}{2}t^2\right| \le C_{\varepsilon,M} \cdot |t|^3. \tag{67}$$

Recall that for any bounded linear operator A acts on a Hilbert space, we set $|A| = \sqrt{A^*A}$. The inequality (67) applied to the eigenvalues of trace class operator with spectrum contained in $[-1+\varepsilon, -1+M]$ yields the following

Lemma 7.9. *Let $\varepsilon, M, C_{\varepsilon,M}$ be as in the inequality (67). For any self-adjoint trace class operator A whose spectrum $\sigma(A)$ satisfies $\sigma(A) \subset [-1+\varepsilon, -1+M]$, we have*

$$\log\det(1+A) \le \mathrm{tr}(A) - \frac{1}{2}\mathrm{tr}(A^2) + C_{\varepsilon,M}\mathrm{tr}(|A|^3). \tag{68}$$

Proof. The lemma is an immediate consequence of the inequality (67) and the identity

$$\log\det(1+A) = \sum_{i=1}^{\infty} \log(1+\lambda_i(A)),$$

where $(\lambda_i(A))_{i=1}^{\infty}$ is the sequence of the eigenvalues of A. $\square$

In order to simplify notation, for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^+$, set

$$h = g-1 \geq 0 \text{ and } T_g^+ = \sqrt{h}\Pi\sqrt{h} \geq 0; \tag{69}$$

and for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^-$, set

$$h = g-1 \leq 0 \text{ and } T_g^- = \Pi h \Pi \leq 0. \tag{70}$$

By applying the relation (68), for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$, we have

$$\begin{aligned} \log \mathbb{E}\Psi_g &= \log\det(1+(g-1)\Pi) = \log\det(1+T_g^{\pm}) \\ &\leq \mathrm{tr}(T_g^{\pm}) - \frac{1}{2}\mathrm{tr}((T_g^{\pm})^2) + C_{\varepsilon,M}\mathrm{tr}(|T_g^{+}|^3). \end{aligned} \tag{71}$$

Clearly, the traces $\mathrm{tr}(T_g^+)$ and $\mathrm{tr}(T_g^-)$ are given by the formula:

$$\mathrm{tr}(T_g^{\pm}) = \int_E h(x)\Pi(x,x)d\mu(x). \tag{72}$$

Recall that the inner product on the space of Hilbert-Schmidt operators is defined by the formula

$$\langle a, b\rangle_{HS} = \mathrm{tr}(ab^*).$$

Lemma 7.10. *For any $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$, we have*

$$\mathrm{tr}((T_g^{\pm})^2) = \int_E h(x)^2\Pi(x,x)d\mu(x) - \frac{1}{2}V(g). \tag{73}$$

Proof. If $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^+$, then

$$\mathrm{tr}((T_g^+)^2) = \mathrm{tr}(\sqrt{h}\Pi h \Pi\sqrt{h}) = \mathrm{tr}(\Pi h \Pi h) = \langle \Pi h, h\Pi\rangle_{HS}. \tag{74}$$

Note that

$$\|\Pi h\|_{HS}^2 = \|h\Pi\|_{HS}^2 = \int_E h(x)^2\Pi(x,x)d\mu(x). \tag{75}$$

By (49), we have

$$\begin{aligned} V(g) &= \|[g,\Pi]\|_{HS}^2 = \|[h,\Pi]\|_{HS}^2 = \|h\Pi - \Pi h\|_{HS}^2 \\ &= \|h\Pi\|_{HS}^2 + \|\Pi h\|_{HS}^2 - 2\langle h\Pi, \Pi h\rangle. \end{aligned} \tag{76}$$

Combining (74), (75) and (76), we complete the proof of the desired identity (73) for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^+$.

The argument for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^-$ is completely the same, since we have

$$\mathrm{tr}((T_g^-)^2) = \mathrm{tr}(\Pi f \Pi f \Pi) = \mathrm{tr}(\Pi f \Pi f).$$

$\square$

Lemma 7.11. *For any $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^\pm$, we have*

$$\mathrm{tr}(|T_g^\pm|^3) \leq L(g) = \int_E |g(x) - 1|^3 \Pi(x,x) d\mu(x). \tag{77}$$

Proof. First, let $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^+$. Recall the definition of h and T_g^+ in (69). By the elementary operator inequality

$$\sqrt{h}\Pi h \Pi h \Pi \sqrt{h} \leq \sqrt{h}\Pi h^2 \Pi \sqrt{h},$$

we get

$$\mathrm{tr}(|T_g^+|^3) = \mathrm{tr}(\sqrt{h}\Pi h \Pi h \Pi \sqrt{h}) \leq \mathrm{tr}(\sqrt{h}\Pi h^2 \Pi \sqrt{h}) = \|\sqrt{h}\Pi h\|_{HS}^2. \tag{78}$$

Since

$$\begin{aligned} \|\sqrt{h}\Pi h\|_{HS}^2 &= \mathrm{tr}(\sqrt{h}\Pi h^2 \Pi \sqrt{h}) = \mathrm{tr}(\Pi h^{3/2} h^{1/2} \Pi h) \\ &= \langle \Pi h^{3/2}, h \Pi h^{1/2} \rangle_{HS} \leq \|\Pi h^{3/2}\|_{HS} \|h \Pi h^{1/2}\|_{HS} \\ &= \|\Pi h^{3/2}\|_{HS} \|\sqrt{h}\Pi h\|_{HS}, \end{aligned}$$

we also have

$$\|\sqrt{h}\Pi h\|_{HS}^2 \leq \|\Pi h^{3/2}\|_{HS}^2 = \mathrm{tr}(\Pi h^3 \Pi) = \mathrm{tr}(h^3 \Pi) = L(g). \tag{79}$$

Combining inequalities (78) and (79), we obtain the desired inequality (77) for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^+$.

The proof of the inequality (77) for $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^-$ is similar just by noting that in this case, $|T_g^-|^3 = -\Pi h \Pi = \Pi |h| \Pi$ and

$$\begin{aligned} \mathrm{tr}(|T_g^-|^3) &= \mathrm{tr}(\Pi |h| \Pi |h| \Pi |h| \Pi) = \mathrm{tr}(\sqrt{|h|}\Pi |h| \Pi |h| \Pi \sqrt{|h|}) \\ &\leq \mathrm{tr}(\sqrt{|h|}\Pi |h|^2 \Pi \sqrt{|h|}). \end{aligned}$$

$\square$

Conclusion of the proof of Lemma 7.7. It suffices to establish (63) when $g \in \mathscr{A}_3^{\varepsilon,M}(\Pi)^{\pm}$. An application of (67) yields that

$$\left| \int_E \left(\log g(x) - h(x) + \frac{h(x)^2}{2} \right) \Pi(x,x) d\mu(x) \right| \leq C_{\varepsilon,M} L(g). \tag{80}$$

It follows that

$$\begin{aligned}
\log \mathbb{E}\widetilde{\Psi}_g =& \log \mathbb{E}\Psi_g - \mathbb{E}S_{\log g} \\
\leq & \mathrm{tr}(T_g^{\pm}) - \frac{1}{2}\mathrm{tr}((T_g^{\pm})^2) + C_{\varepsilon,M}\mathrm{tr}(|T_g^{\pm}|^3) - \mathbb{E}S_{\log g} \\
\leq & \int_E h(x)\Pi(x,x)d\mu(x) - \frac{1}{2}\int_E h(x)^2 \Pi(x,x) d\mu(x) + \frac{1}{4}V(g) \\
& + C_{\varepsilon,M}L(g) - \int_E \log g(x) \Pi(x,x) d\mu(x) \\
\leq & 2C_{\varepsilon,M}L(g) + \frac{1}{4}V(g) = C'_{\varepsilon,M}(L(g) + V(g)).
\end{aligned}$$

□

Proposition 7.12. *Given* $0 < \varepsilon \leq 1$ *and* $M \geq 1$, *there exists a constant* $C_{\varepsilon,M} > 0$ *such that if* $g_1, g_2 \in \mathscr{A}_3^{\varepsilon,M}(\Pi)$, *then*

$$\left(\mathbb{E}|\widetilde{\Psi}_{g_1} - \widetilde{\Psi}_{g_2}| \right)^2 \leq \mathbb{E}|\widetilde{\Psi}_{g_2}|^2 \cdot \left[\exp\left(C_{\varepsilon,M}\big(L(g_1/g_2) + V(g_1/g_2)\big) \right) - 1 \right]. \tag{81}$$

Proof. Let g_1, g_2 be as in the proposition. Set $g := (g_1/g_2)^2$. Applying Proposition 7.6 to the function g yields

$$\mathbb{E}\widetilde{\Psi}_g \leq \exp\left(C_{\varepsilon,M}\Big(L(g) + V(g)\Big)\right) \leq \exp\left(C'_{\varepsilon,M}\Big(L(g_1/g_2) + V(g_1/g_2)\Big)\right).$$

By multiplicativity, we have

$$\mathbb{E}|\widetilde{\Psi}_{g_1} - \widetilde{\Psi}_{g_2}| = \mathbb{E}\Big(|\widetilde{\Psi}_{g_1/g_2} - 1||\widetilde{\Psi}_{g_2}| \Big) \leq \left(\mathbb{E}|\widetilde{\Psi}_{g_2}|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}|\widetilde{\Psi}_{g_1/g_2} - 1|^2 \right)^{\frac{1}{2}}.$$

Since $\mathbb{E}\widetilde{\Psi}_{g_1/g_2} \geq 1$, we have

$$\mathbb{E}|\widetilde{\Psi}_{g_1/g_2} - 1|^2 \leq \mathbb{E}|\widetilde{\Psi}_{g_1/g_2}|^2 - 1 = \mathbb{E}\widetilde{\Psi}_g - 1.$$

Combining the above inequalities, we obtain Proposition 7.12. □

Slightly abusing notation, we keep the notation $\mathscr{T}$ for the induced topology defined by (53) on $\mathscr{A}_3^{\varepsilon,M}(\Pi)$. As an immediate consequence of Proposition 7.12, we have

Corollary 7.13. *The two mappings from* $\mathscr{A}_3^{\varepsilon,M}(\Pi)$ *to* $L^1(\mathrm{Conf}(E),\mathbb{P}_\Pi)$ *defined by*

$$g \to \widetilde{\Psi}_g, \quad g \to \overline{\Psi}_g$$

are continuous with respect to the topology $\mathscr{T}$ *on* $\mathscr{A}_3^{\varepsilon,M}(\Pi)$.

Proof of Proposition 7.2. The proof follows the proof of Corollary 4.11 in [1], the rôle of Proposition 4.8 of [1] played here by Corollary 7.13. Indeed, let g be a function satisfying the assumption (50). Taking g_n as in Lemma 7.3, we obtain the convergence of Π^{g_n} to Π^g in the space of locally trace class operators and hence the weak convergence of $\mathbb{P}_{\Pi^{g_n}}$ to $\mathbb{P}_{\Pi^g}$ in the space of probability measures on $\mathrm{Conf}(E)$. By assumption, $g_n - 1$ is compactly supported, so by Proposition 2.1 of [2], we have

$$\mathbb{P}_{\Pi^{g_n}} = \overline{\Psi}_{g_n} \cdot \mathbb{P}_\Pi.$$

By Corollary 7.13, $\overline{\Psi}_{g_n} \to \overline{\Psi}_g$ in $L^1(\mathrm{Conf}(E),\mathbb{P}_\Pi)$, so we have

$$\overline{\Psi}_{g_n} \cdot \mathbb{P}_\Pi \to \overline{\Psi}_g \cdot \mathbb{P}_\Pi$$

weakly in the space of probability measures on $\mathrm{Conf}(E)$, whence

$$\mathbb{P}_{\Pi^g} = \overline{\Psi}_g \cdot \mathbb{P}_\Pi.$$

The proof Proposition 7.2 is complete. □

8 Appendix

Our aim here is to show that Palm measures of different orders are *mutually singular* for a point process rigid in the sense of Ghosh [8], Ghosh-Peres [9].

Let E be a complete metric space, and let $\mathbb{P}$ be a probability measure on $\mathrm{Conf}(E)$ admitting correlation measures of all orders; the k-th correlation measure of $\mathbb{P}$ is denoted by ρ_k. Given $B \subset E$ a bounded Borel subset, let $\mathfrak{F}(E\setminus B)$ be the sigma-algebra generated by all events of the form $\{\#_C = n\}$ with $C \subset E \setminus B$ bounded and Borel, $n \in \mathbb{N}$, and let $\mathfrak{F}^{\mathbb{P}}(E\setminus B)$ be the completion of $\mathfrak{F}(E \setminus B)$ with respect to $\mathbb{P}$. We can canonically identify $\mathrm{Conf}(E)$ with $\mathrm{Conf}(B)\times \mathrm{Conf}(E\setminus B)$. Then in this identification, the events in $\mathfrak{F}(E\setminus B)$ have the form

$$\mathrm{Conf}(B) \times A,$$

where $A \subset \mathrm{Conf}(E \setminus B)$ is a measurable subset. By definition, assume that $\mathscr{X} \in \mathfrak{F}(E \setminus B)$, and let $(p_1,\dots,p_k) \in B^k$ be any k-tuple of distinct points, then $\mathfrak{X} \in \mathscr{X}$ if and only if $\mathfrak{X} \cup \{p_1,\dots,p_k\} \in \mathscr{X}$. Recall that a point process with distribution $\mathbb{P}$ on $\mathrm{Conf}(E)$ is said to be rigid if for any bounded Borel subset $B \subset E$, the function $\#_B$ is $\mathfrak{F}^{\mathbb{P}}(E \setminus B)$-measurable.

Proposition 8.1. *Let $B \subset E$ be a bounded Borel subset. Assume that the function $\#_B$ is $\mathfrak{F}^{\mathbb{P}}(E \setminus B)$-measurable. Then, for any $k, l \in \mathbb{N}$, $k \neq l$, for ρ_k-almost any k-tuple $(p_1, \dots, p_k) \in B^k$ and ρ_l-almost any l-tuple $(q_1, \dots, q_l) \in B^l$, the reduced Palm measures $\mathbb{P}^{p_1,\dots,p_k}$ and $\mathbb{P}^{q_1,\dots,q_l}$ are mutually singular.*

Proof. For a nonnegative integer n, let

$$\mathscr{C}_n = \{\mathfrak{X} \in \mathrm{Conf}(E) : \#_B(\mathfrak{X}) = n\}.$$

By assumption, the function $\#_B$ is $\mathfrak{F}^{\mathbb{P}}(E \setminus B)$-measurable. Take a sequence $\mathscr{X}_n$ of disjoint $\mathfrak{F}(E \setminus B)$-measurable subsets of $\mathrm{Conf}(E)$ such that for any nonnegative integer n we have

$$\mathbb{P}(\mathscr{X}_n \Delta \mathscr{C}_n) = 0.$$

Set

$$\mathscr{Y} = \bigcup_{n \geq k} \mathscr{X}_n \cap \mathscr{C}_{n-k};$$

$$\mathscr{Z} = \bigcup_{n \geq l} \mathscr{X}_n \cap \mathscr{C}_{n-l}.$$

The sets $\mathscr{Y}$ and $\mathscr{Z}$ are disjoint by construction.

Claim: For ρ_k-almost any k-tuple $(p_1, \dots, p_k)$ and ρ_l-almost any l-tuple $(q_1, \dots, q_l)$ we have

$$\mathbb{P}^{p_1,\dots,p_k}(\mathscr{Y}) = 1, \quad \mathbb{P}^{q_1,\dots,q_l}(\mathscr{Z}) = 1.$$

Indeed, by definition of reduced Palm measures (17), for any non-negative Borel function $u : \mathrm{Conf}(E) \times E^k \to \mathbb{R}$, we have

$$\begin{aligned} &\int\limits_{\mathrm{Conf}(E)} \sum_{z_1,\dots,z_k \in \mathcal{Z}}^{*} u(\mathcal{Z}; z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\ &= \int\limits_{E^k} \rho_k(dp_1 \dots dp_k) \int\limits_{\mathrm{Conf}(E)} u(\mathfrak{X} \cup \{p_1, \dots, p_k\}; p_1, \dots, p_k) \mathbb{P}^{p_1,\dots,p_k}(d\mathfrak{X}), \end{aligned} \tag{82}$$

where $\sum^{*}$ denotes the sum over k-tuples of distinct points $z_1, \dots, z_k$ in $\mathcal{Z}$.

For any $n \geq k$, substituting the function

$$u_n(\mathcal{Z}; z_1, \dots, z_k) = \mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_n}(\mathcal{Z}) \cdot \mathbb{1}_{B^k}(z_1, \dots, z_k)$$

into (82), we get

$$\begin{aligned} &\int\limits_{\mathrm{Conf}(E)} \mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_n}(\mathcal{Z}) \sum_{z_1,\dots,z_k \in Z}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\ &= \int\limits_{B^k} \rho_k(dp_1 \dots dp_k) \int\limits_{\mathrm{Conf}(E)} \mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_n}(\mathfrak{X} \cup \{p_1, \dots, p_k\}) \mathbb{P}^{p_1,\dots,p_k}(d\mathfrak{X}). \end{aligned} \tag{83}$$

Recall that by construction, $\mathscr{X}_n \in \mathfrak{F}(E \setminus B)$, hence *for all* $p_1, \dots, p_k \in B$, we have

$$\begin{aligned}
&\mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_n}(\mathcal{X} \cup \{p_1, \dots, p_k\}) \\
=&\mathbb{1}_{\mathscr{X}_n}(\mathcal{X} \cup \{p_1, \dots, p_k\}) \cdot \mathbb{1}_{\mathscr{C}_n}(\mathcal{X} \cup \{p_1, \dots, p_k\}) \\
=&\mathbb{1}_{\mathscr{X}_n}(\mathcal{X}) \cdot \mathbb{1}_{\mathscr{C}_{n-k}}(\mathcal{X}) = \mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_{n-k}}(\mathcal{X}).
\end{aligned}$$

Substituting the above equality into (83), we get

$$\begin{aligned}
&\int\limits_{\mathrm{Conf}(E)} \mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_n}(\mathcal{Z}) \sum_{z_1, \dots, z_k \in \mathcal{Z}}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\
=&\int\limits_{B^k} \mathbb{P}^{p_1, \dots, p_k}(\mathscr{X}_n \cap \mathscr{C}_{n-k}) \rho_k(dp_1 \dots dp_k).
\end{aligned} \tag{84}$$

Summing up the terms on the left hand side of (84) for $n \geq k$, we obtain the expression

$$\begin{aligned}
&\sum_{n=k}^{\infty} \int\limits_{\mathrm{Conf}(E)} \mathbb{1}_{\mathscr{X}_n \cap \mathscr{C}_n}(\mathcal{Z}) \sum_{z_1, \dots, z_k \in \mathcal{Z}}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\
=&\sum_{n=k}^{\infty} \int\limits_{\mathrm{Conf}(E)} \mathbb{1}_{\mathscr{C}_n}(\mathcal{Z}) \sum_{z_1, \dots, z_k \in \mathcal{Z}}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\
=&\sum_{n=0}^{\infty} \int\limits_{\mathrm{Conf}(E)} \mathbb{1}_{\mathscr{C}_n}(\mathcal{Z}) \sum_{z_1, \dots, z_k \in \mathcal{Z}}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\
=&\int\limits_{\mathrm{Conf}(E)} \sum_{z_1, \dots, z_k \in \mathcal{Z}}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) \mathbb{P}(d\mathcal{Z}) \\
=&\int\limits_{E^k} \mathbb{1}_{B^k}(p_1, \dots, p_k) \rho_k(dp_1 \dots dp_k) = \rho_k(B^k),
\end{aligned} \tag{85}$$

where we used the fact that if $n = 0, \dots, k-1$, then

$$\forall \mathcal{Z} \in \mathscr{C}_n, \sum_{z_1, \dots, z_k \in \mathcal{Z}}^{*} \mathbb{1}_{B^k}(z_1, \dots, z_k) = 0.$$

Similarly, summing up the terms on the right hand side of (84) for $n \geq k$, we obtain the expression

$$\begin{aligned}
&\sum_{n=k}^{\infty} \int\limits_{B^k} \mathbb{P}^{p_1, \dots, p_k}(\mathscr{X}_n \cap \mathscr{C}_{n-k}) \rho_k(dp_1 \dots dp_k) \\
=&\int\limits_{B^k} \mathbb{P}^{p_1, \dots, p_k}\left(\bigcup_{n \geq k} \mathscr{X}_n \cap \mathscr{C}_{n-k} \right) \rho_k(dp_1 \dots dp_k) \\
=&\int\limits_{B^k} \mathbb{P}^{p_1, \dots, p_k}\left(\mathscr{Y} \right) \rho_k(dp_1 \dots dp_k).
\end{aligned} \tag{86}$$

By (84),

$$\rho_k(B^k) = \int\limits_{B^k} \mathbb{P}^{p_1,\dots,p_k}(\mathscr{Y})\,\rho_k(dp_1\dots dp_k). \tag{87}$$

The equality (87) immediately implies that

$$\mathbb{P}^{p_1,\dots,p_k}(\mathscr{Y}) = 1, \text{ for } \rho_k\text{-almost any } k\text{-tuple } (p_1,\dots,p_k) \in B^k.$$

The same argument yields that

$$\mathbb{P}^{q_1,\dots,q_l}(\mathscr{Z}) = 1, \text{ for } \rho_l\text{-almost any } l\text{-tuple } (q_1,\dots,q_l) \in B^l.$$

The claim is proved, and Proposition 8.1 is proved completely. □

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